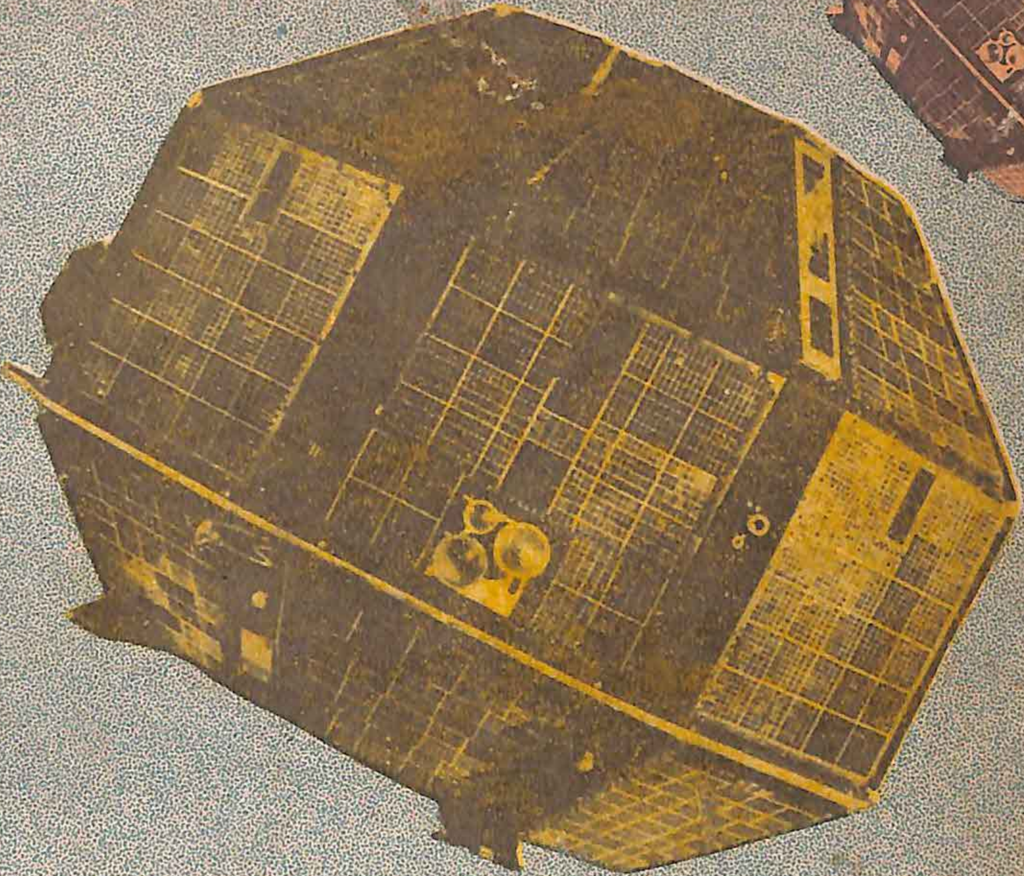
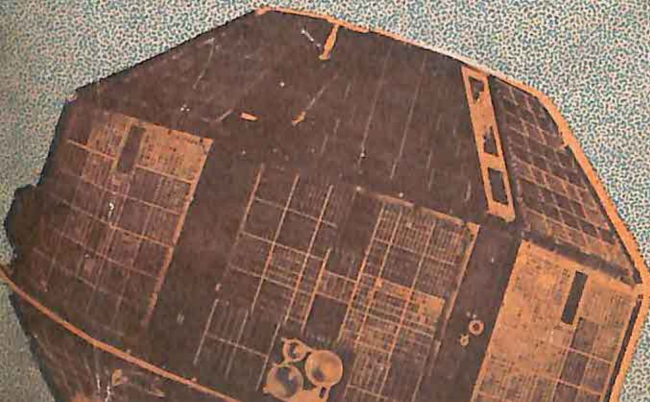


# MATHEMATIC



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# MATHEMATICS

A TEXTBOOK FOR SECONDARY SCHOOLS

## PART II

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# Mathematics

## A TEXTBOOK FOR SECONDARY SCHOOLS

## PART II



राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

**First Edition**
*January 1986*
*Magha 1907*

**Reprinted**
*November 1986*
*Agrahayana 1908*

**P.D. 170 T–RP**





**Rs. 11.05**

Published at the Publication Department by the Secretary, National Council of Educational Research and Training, Sri Aurobindo Marg, New Delhi 110016, and printed at Saraswati Offset Printers, Saraswati House, A-5/11, Naraina Industrial Area, New Delhi 110028.

# Foreword

Work on the present textbook was jointly initiated by the National Council of Educational Research and Training (NCERT) and the Central Board of Secondary Education (CBSE) in March 1985. The authors developed draft materials under the guidance of Prof. Mohan Lal, Convener of the group. A National Seminar-cum-Workshop was organised in June 1985 to review the draft textual materials with a view to provide suggestions for improving the materials. The authors revised their textual content in the light of the suggestions made during the Workshop. The present textbook is the outcome of these efforts.

The National Council is thankful to the Central Board of Secondary Education, particularly to the Chairman, Fr. T.V. Kunnunkal, and the Director (Academic), Dr. K.D. Sharma, for initiating the process of curriculum and textbook revision in mathematics, for providing inputs—both expertise and materials—to the National Seminar-cum-Workshop, and for active collaboration throughout the development of this new textbook. My special thanks are due to the Convener, Prof. Mohan Lal, and other members of the CBSE Committee of Courses in Mathematics. I must also express my thanks to Prof. A.K. Jalaluddin, Joint Director, NCERT and to Prof. B. Ganguly, Head of the Department of Education in Science and Mathematics, NCERT, who have borne much of the burden in organising the revision of the new science and mathematics syllabi and the preparation of the textbooks. I am grateful to the authors of this book for preparing and finalising the draft chapters on the basis of the suggestions received.

My appreciation and thanks are due also to Prof. K.V. Rao, Prof. S.C. Das, Shri G.S. Baderia, Dr. B. Deokinandan, Dr. Ram Autar, Dr. (Smt.) Surja Kumari, Shri Ishwar Chandra and Shri Mahendra Shanker, all members of the Mathematics Group of the Department of Education in Science and Mathematics, for their valuable contribution and for seeing the book through the press.

Suggestions for further improvement of the book will be most welcome.

P.L. MALHOTRA
*Director*
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New Delhi
*January 1986*

# Preface

Consequent upon the decision to have only one stream in mathematics and science at the secondary stage in schools affiliated to the Central Board of Secondary Education (CBSE), the Central Board's Committee of Courses in Mathematics was asked to frame a new syllabus in mathematics for Classes IX and X, and subsequently to get the new textbooks written.

The Committee sought the guidance of eminent educationists and mathematicians, including Prof. Shanti Narayan, Prof. J.N. Kapur, Prof. S.D. Chopra, Prof. M.K. Singhal, and others from the NCERT and the University of Delhi and its colleges. The consensus was that more emphasis should be laid on the teaching of mathematics relevant to the day-to-day needs of the citizens, and less on proof-making in geometry. Accordingly, the weightage on geometry was decreased, and on commercial mathematics, increased.

Steps were initiated to develop a textbook for Class IX on these lines. With the help of authors from various organisations like the NCERT, the Delhi University and its colleges, government schools, private schools and Kendriya Vidyalayas, the textbook was developed and published by NCERT as a joint venture of the CBSE and the NCERT, in January 1985.

Subsequently as Convener of the Committee for developing textbook for Class X, I selected a team of authors from various organisations like the NCERT, the Delhi University and its colleges, government schools, private schools and Kendriya Vidyalayas. (Their names are given in a separate list). The whole group was divided into four sub-groups which met several times in the CBSE and in the NCERT and prepared the draft manuscript. This draft was discussed in a national workshop organised by the NCERT and the suggestions made in the workshop were incorporated to the extent possible.

I am indebted to all the authors who worked hard for developing the manuscript. Our thanks are due to Shri J.N. Rampal and Dr. Chandrakant Bhardwaj who undertook the work of preparing the Hindi version of the Class IX and Class X textbooks. I am grateful to Dr. P.L. Malhotra, Director, NCERT, Prof. A.K. Jalaluddin, Joint Director, NCERT and Prof. B. Ganguly, Head, Department of Education in Science and Mathematics, for their encouragement and help in completing this task and to Fr. T.V. Kunnunkal, Chairman, CBSE, Dr. K.D. Sharma, Director (Academic) and other staff

of the CBSE. I express my sincere thanks to Prof. K.V. Rao and the other members of the Mathematics Group of the Department of Education in Science and Mathematics, NCERT, for preparing the final copy of the manuscript and seeing the book through the press. I am personally indebted to them all.

In this book we have made an effort to cater to the needs to the average student, but there are discussions which may be challenging to the brighter students as well. This approach had to be followed because this book is meant for each and every student of Class X who has to study the subject compulsorily. One of the important features of the book is that numerous worked-out examples have been given for a better understanding of the concepts and principles by the students, and these are followed by exercises which give enough practice for mastering the concepts. Another important feature is that except where a formal proof was considered useful, the geometrical results have been exhibited through the activity approach, including paper-folding, tracing and cutting. It is hoped that the teacher will do some of these activities in the classroom and encourage the students to do more at home.

The authors would like to request the teachers and the pupils to go through the pages of the text in detail and not remain content with solving the problems only. No book is the last word on the subject and, therefore, we shall appreciate any constructive criticism of the book within the broad framework in which we have worked. Suggestions for improvement of the text will be most welcome.

MOHAN LAL
*Principal*
PGDAV College, New Delhi
and
*Convener*
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*January 1986*

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# Contents

CHAPTER 1

# Linear Inequations and Their Applications

## 1.1 System of Inequations

In the previous class, we have studied how to solve and graph a system of equations in two variables. Also we have studied how to determine solutions of a linear inequation in two variables. If we have two or more inequations in two variables, we can find the solutions that satisfy all the inequations. We call those inequations a **system of linear inequations.** The solution of the system can be obtained by finding the intersection of the solution sets of inequations forming the system. This can be done algebraically as well as by graphing the inequations on the same plane and determining their common points. However we will restrict ourselves to graphical solutions only. To illustrate the procedure, we consider the following examples:

**Example 1 :** Obtain the solution of the system

$$\begin{cases} -2x + y \leqslant 3 \\ 2x + y > 4 \end{cases}$$

**Solution :** To determine the solution of the given system, we graph each of the inequations of the system, on the same coordinate plane.

We have read in class IX how to graph a linear inequation. Can you graph the inequation $-2x + y \leqslant 3$ ?

Graph of this inequation is shown in figure 1.1. The solution set of $-2x + y \leqslant 3$ consists of all the points in the region below the graph of $-2x + y = 3$. The line $-2x + y = 3$ is a part of the graph. Similarly, you can plot the graph of $2x + y > 4$ on the same coordinate plane. The solution set of $2x + y > 4$ consists of the region above the graph of $2x + y = 4$. We may note that the line $2x + y = 4$ is not a part of this graph.

Thus, the intersection (common points) of the two shaded regions represents the intersection of the two solution sets and contains those and only those points which satisfy both the inequations. Some of the common points of the solution set are (2,2), (3,3), (3,4) and so on. In fact, all the points in the double shaded region are the members of the solution set.

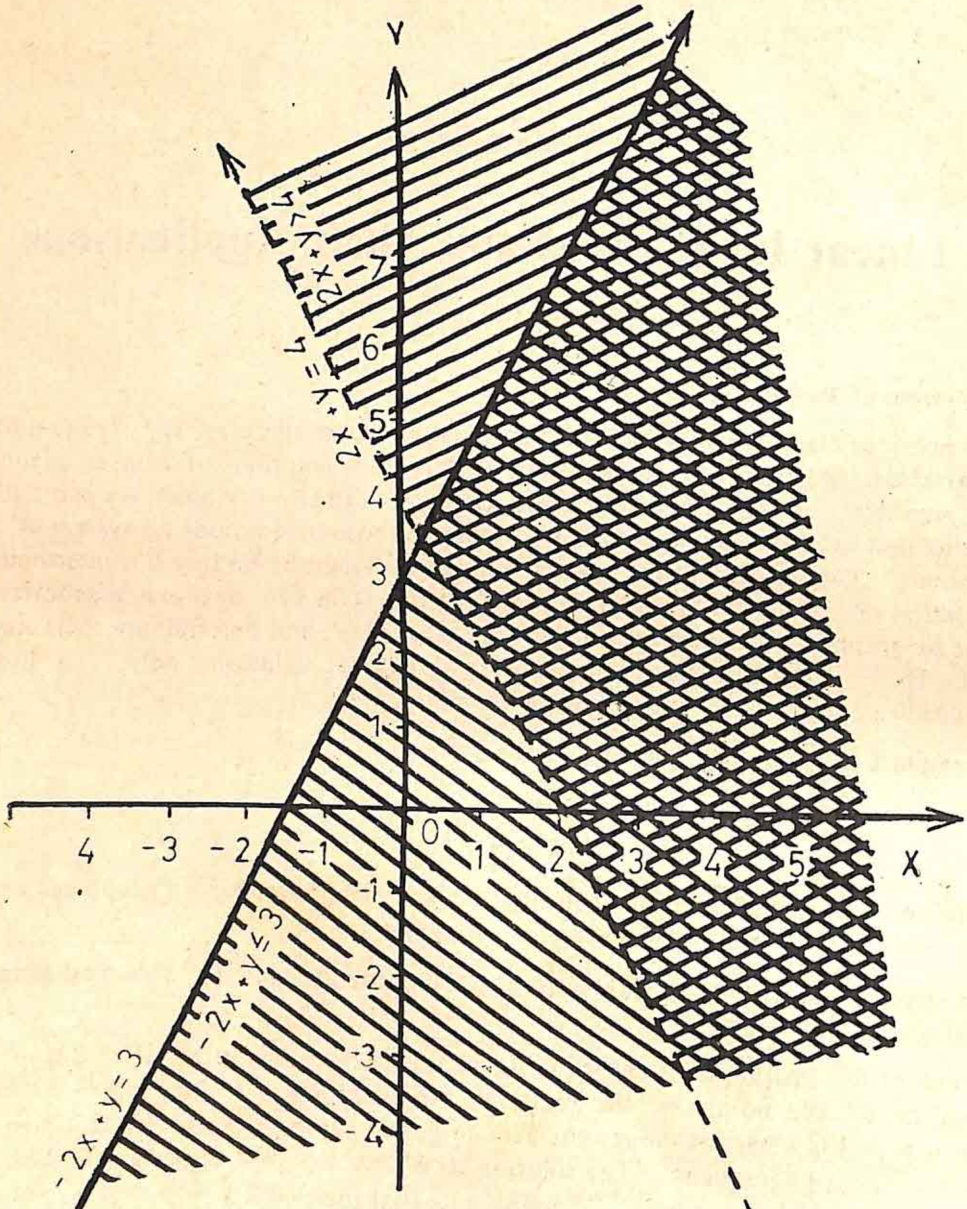



Fig. 1.1

**Example 2 :** Solve the system of inequations

$$\begin{cases} 3x - y < 4 \\ 3x - y > -2 \end{cases}$$

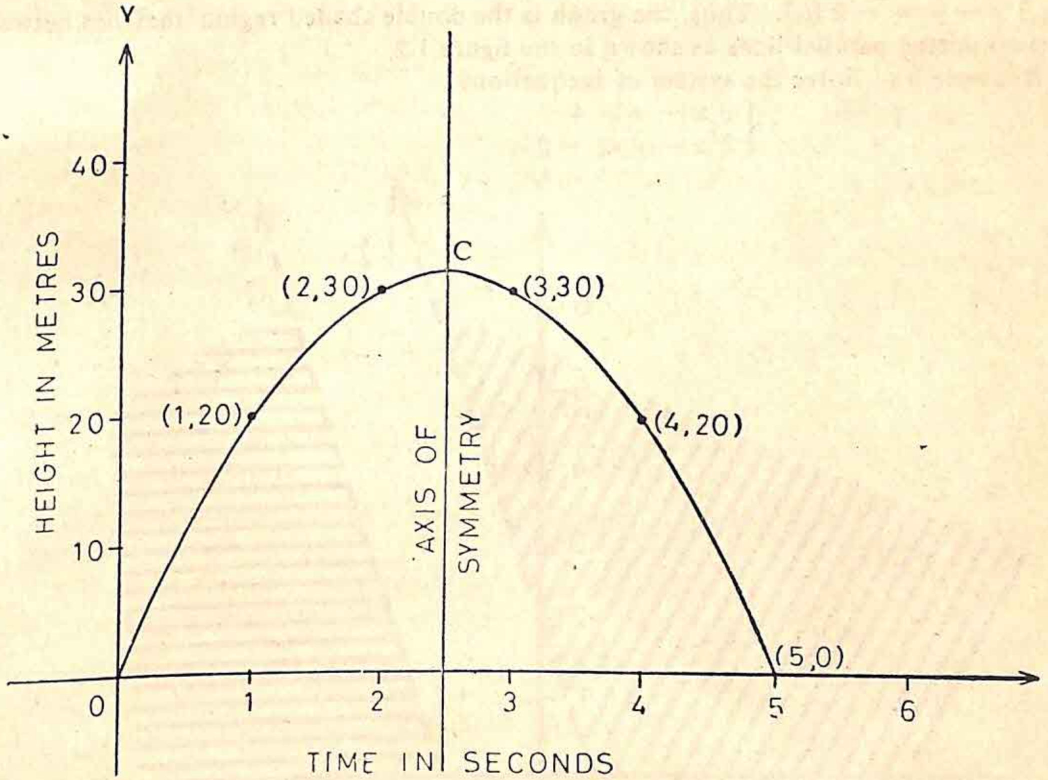



Fig. 3.1

the $x$-axis are the zeros of the polynomial $-5x^2 + 25x$. From the grapn we find that 0 and 5 are the zeros of the polynomial.

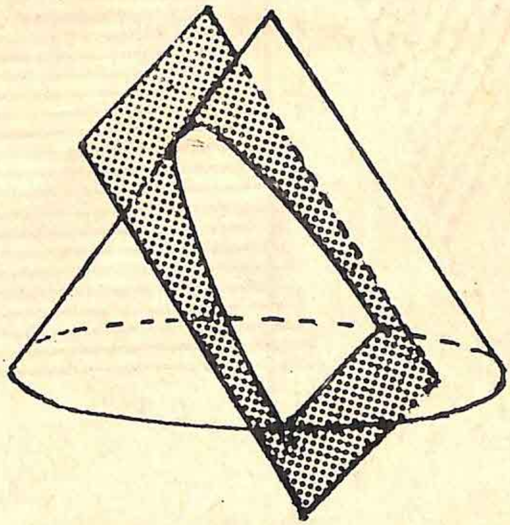
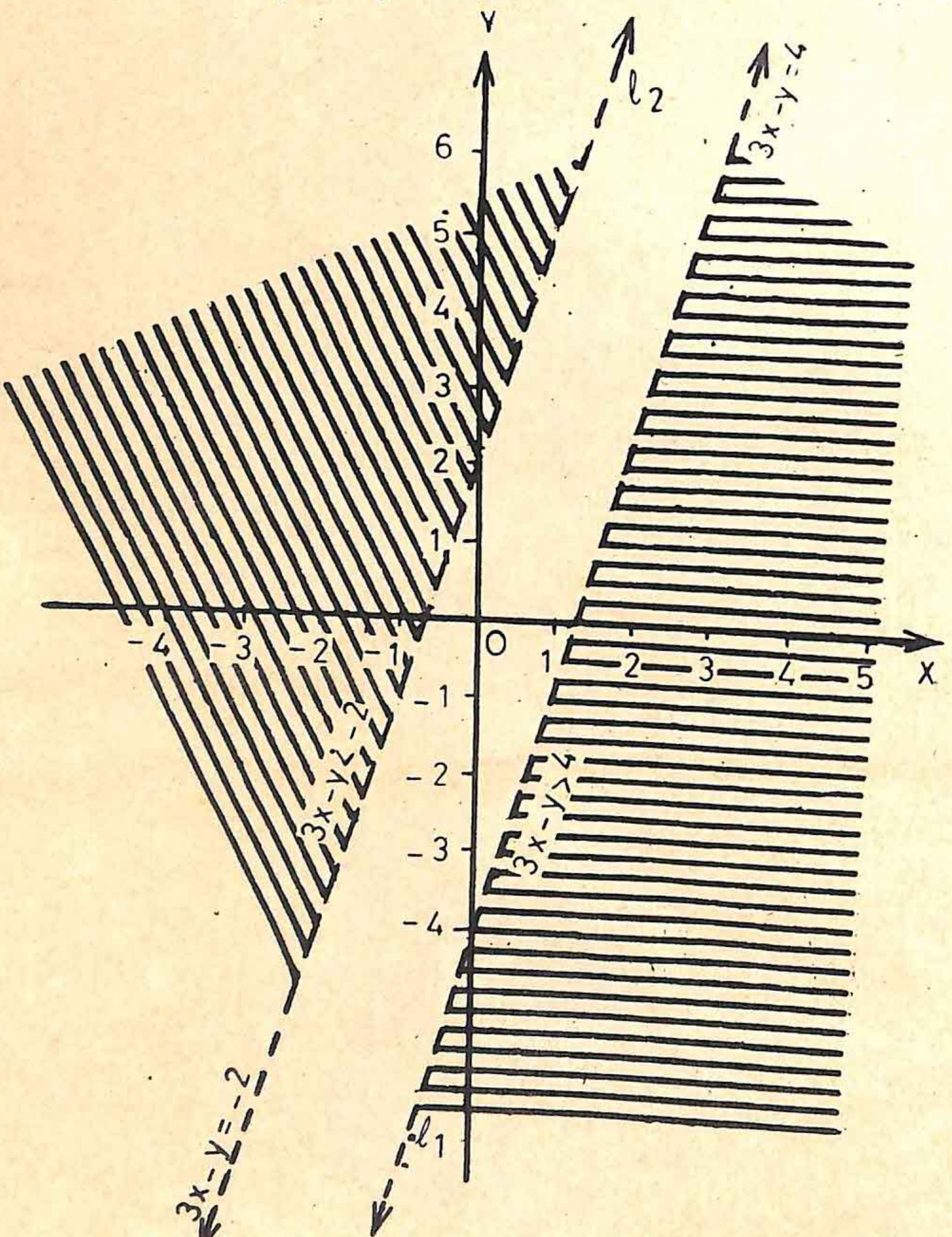


Fig. 3.2

line $3x - y = -2$ ($l_2$). Thus, the graph is the double shaded region that lies between the two dotted parallel lines as shown in the figure 1.2.

**Example 3 :** Solve the system of inequations

$$\begin{cases} 3x - y > 4 \\ 3x - y < -2 \end{cases}$$




Fi . 1.3

***Solution :*** *This* system is similar to the one given in example 2. However, the inequality symbol in each inequation has been reversed.

The graph of $3x - y > 4$ is the shaded region below $l_1$ as shown in figure 1.3.

The graph of $3x - y < -2$ is the shaded region above the line $l_2$. It may be seen that $l_1$ and $l_2$ are parallel. Hence we observe that there is no common shaded region. Therefore, the system has no solution.

**Example 4 :** Graph each inequation in the following system on the same plane. Show the solution set of the system as the set of points in a triply shaded region.

$$\begin{cases} y \leqslant x \\ y \geqslant 0 \\ x \leqslant 4 \end{cases}$$

**Solution :** The solution set of $y \leqslant x$ consists of all the points on $y = x$ and in the shaded region below it.

The solution set of $y \geqslant 0$ consists of all the points on the x-axis and in the entire region above it.

The solution set of $x \leqslant 4$ consists of all the points on the line $x = 4$ and in the entire region to the left of it as shown in figure 1.4.

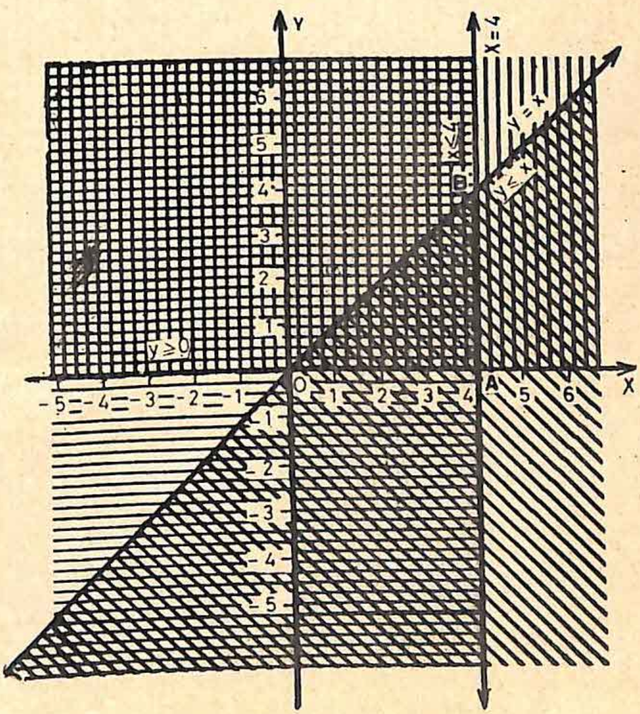



**Fig. 1.4**

The intersection of these three sets is the shaded triangular region $OAB$ which include all the points of the boundary as well as the interior.

**Example 5 :** Graph the following system of inequations

$$\begin{cases} 3x + 2y \leqslant 18 \\ x + 2y \leqslant 10 \\ x \geqslant 0 \\ y \geqslant 0 \end{cases}$$

Find the ordered pairs representing each of the vertices of the region so formed.

**Solution :** Draw the graphs of the inequations of the given system on the same $XOY$-plane. The solution set consists of all the points of the quadrilateral region $OABC$

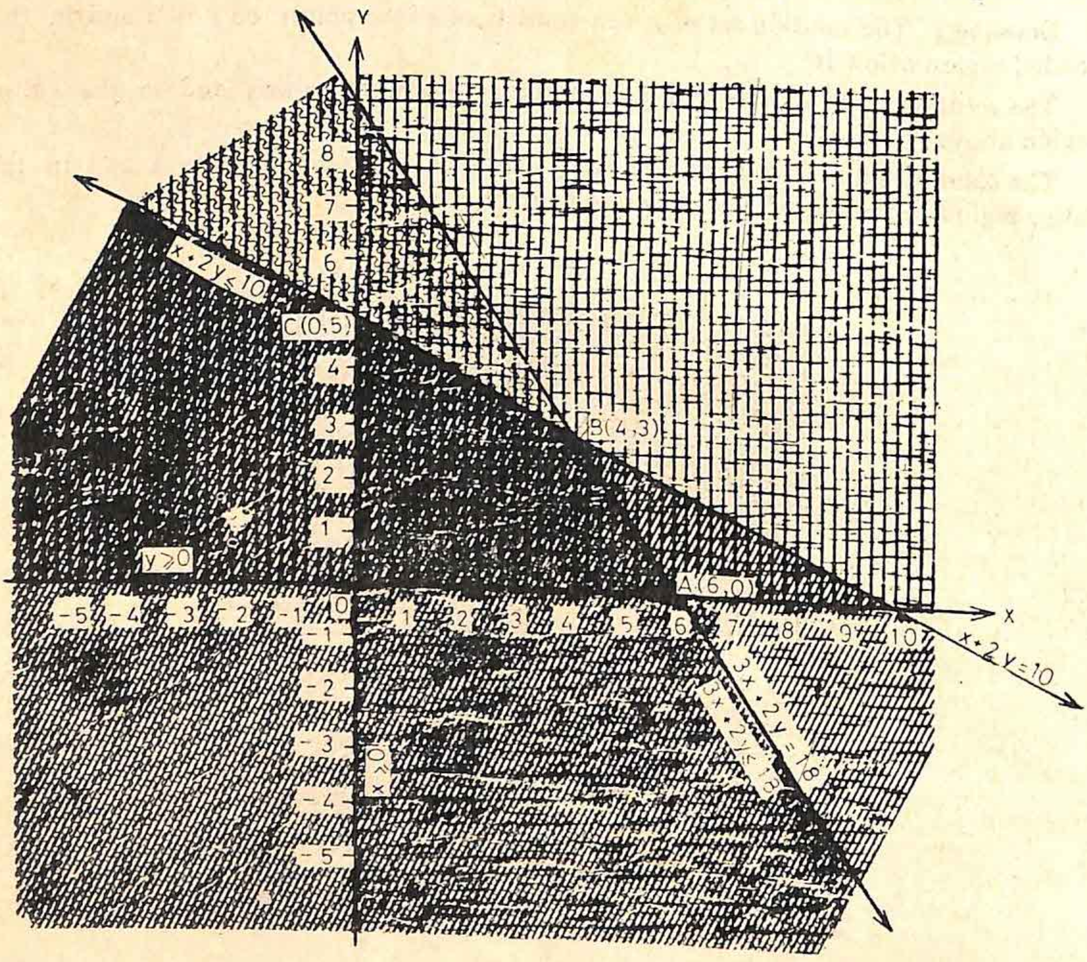



Fig. 1.5

which include the points of the boundary as well as the interior as shown in figure 1.5. *Thus, the region representing the solution set of the given system of inequations is the quadrilateral OABC with vertices (0,0), (6,0), (4,3)* and (0,5) respectively along with its interior.

Till now we have studied a system consisting of inequations only. In the following example, we shall now be studying the system consisting of one equation and one inequation. If the system consists of several inequations and an equation, then the method of finding the solution set is the same as in this example.

**Example 6 :** Solve the following system graphically :

$$\begin{cases} y = x \\ x + y \geqslant 2 \end{cases}$$

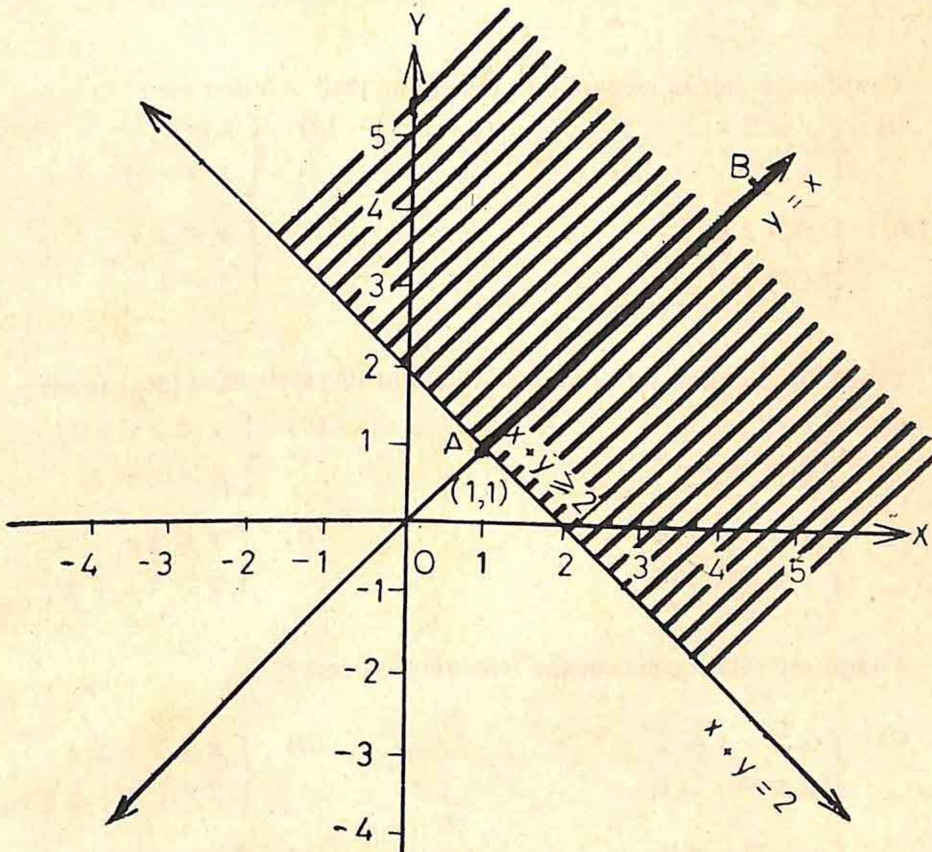



**Fig. 1.6**

**Solution :** The thick portion of the line $AB$ including the point $A$ (1,1) (ray $AB$) is the graph of the solution set, as shown in figure 1.6.

However we can verify the solution algebraically also as is done below.

By substituting $x$ for $y$ in $x + y \geqslant 2$, we have

$$2x \geqslant 2 \Rightarrow x \geqslant 1$$

Hence the solution set is

$$\{(x, y) \mid x = y \text{ and } x \geqslant 1\}$$

Thus, the ray $AB$ is the required solution set of the system.

## Exercises 1.1

1. Graph each pair of inequations indicating their solution set :

(i) $\begin{cases} y \geqslant 3x \\ y \leqslant 3 \end{cases}$ (ii) $\begin{cases} y \geqslant x \\ y \geqslant -1 \end{cases}$

(iii) $\begin{cases} y \geqslant 2x \\ x > 1 \end{cases}$ (iv) $\begin{cases} y < 2x \\ x \geqslant 1 \end{cases}$

2. Graph the solution set of each of the following systems of inequations :

(i) $\begin{cases} y < x - 1 \\ y > 1 - x \end{cases}$ (ii) $\begin{cases} y < 2x + 1 \\ y > x - 2 \end{cases}$

(iii) $\begin{cases} y \geqslant 3x - 6 \\ y \leqslant 2x + 4 \end{cases}$ (iv) $\begin{cases} y \leqslant 3x - 3 \\ y \geqslant 2x + 2 \end{cases}$

3. Graph the solution sets of the following systems :

(i) $\begin{cases} 2x - y \leqslant 2 \\ 2x - y \geqslant 0 \end{cases}$ (ii) $\begin{cases} x + 2y \geqslant 4 \\ 2x + 4y \leqslant 8 \end{cases}$

(iii) $\begin{cases} x - 3y \leqslant 6 \\ x - 3y > -6 \end{cases}$ (iv) $\begin{cases} 3x + y > 3 \\ 3x + y \leqslant -3 \end{cases}$

4. Graph the solution set of each of the following systems of inequations :

(i) $\begin{cases} y \geqslant x \\ x \geqslant 0 \\ y \leqslant 4 \end{cases}$ (ii) $\begin{cases} x + y < 0 \\ y > -2 \\ x > -2 \end{cases}$

(iii) $\begin{cases} x + 3y \leqslant 6 \\ y \geqslant 0 \\ x \geqslant 0 \end{cases}$ (iv) $\begin{cases} -x + 2y \leqslant 10 \\ 2x - y \geqslant -4 \\ y \leqslant 8 \end{cases}$

5. Graph the following systems of inequations. In each cases name the geometric figure so formed and find the ordered pairs representing its vertices.

(i) $\begin{cases} x \leqslant 2 \\ y \leqslant 2 \\ x \geqslant 0 \\ y \geqslant 0 \end{cases}$ (ii) $\begin{cases} x \geqslant 1 \\ y \geqslant 1 \\ x \leqslant 4 \\ y \leqslant 3 \end{cases}$

(iii) $\begin{cases} y \geqslant x - 2 \\ x \geqslant y - 1 \\ x \geqslant 2 \\ x \leqslant 4 \end{cases}$ (iv) $\begin{cases} 2x + y \leqslant 9 \\ y \leqslant x \\ y \leqslant 3 \\ y \geqslant 0 \end{cases}$

6. Solve each of the following systems graphically :

(i) $\begin{cases} y = 2x \\ x < 0 \end{cases}$ (ii) $\begin{cases} 2x + 3y + 6 = 0 \\ y \geqslant 0 \end{cases}$

(iii) $\begin{cases} x + y = 0 \\ y < x + 2 \end{cases}$ (iv) $\begin{cases} x - 2y = 2 \\ x + 2y \geqslant 6 \end{cases}$

## 1.2 Linear Programming

As applications of the study of system of linear inequations, an important technique known as linear programming was developed. It consists of maximising or minimising a linear function of several variables, called the objective function, subject to a number of constraints.

A firm frequently requires several components for the manufacture of items it produces, and there are usually several stages for assembly of each item and finally reaching the customer. The company's costs and profits on the item depend on the availability of the components, number of items produced and the manpower requirements for the production of the item. If the relations among the various resources, the

production requirements, the cost, and the profits are all linear, then these activities may be planned in the best possible way by means of **linear programming.**

Linear programming is useful in solving the problems of allocating limited resources among various activities in the best possible way. Linear programming problems involve many variables, but in this section we shall restrict ourselves to problems involving only two variables.

## 1.3 Mathematical Formulation of the Problem

Let us consider the following linear programming problem. A dealer wishes to purchase fans and sewing machines. He has only Rs 5760 to invest and has space for at the most 20 items. A fan costs him Rs 360 and a sewing machine Rs 240. His expectation is that he can sell a fan at a profit of Rs 22 and a sewing machine at a profit of Rs 18.

Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximise his profit ?

Now in this problem we note that

Maximum possible investment = Rs 5760
Maximum storage space = 20 items (fans and sewing machines)
Cost of a fan = Rs 360
Cost of a sewing machine = Rs 240
Possible profit on a fan = Rs 22
Possible profit on a sewing machine = Rs 18

Let $x$ be the number of fans and $y$ be the number of sewing machines that he buys.
Clearly $x \geqslant 0$ and $y \geqslant 0$

Also there are some more restrictions on the dealer regarding the maximum amount that he can spend and number of pieces of fans and sewing machines that he can store.
So, we have :

$$360\,x + 240\,y \leqslant 5760$$
$$x + y \leqslant 20$$

Also the dealer wants to invest in such a way so as to maximise his profit

$$P = 22\,x + 18\,y$$

Mathematically, the given problem now reduces to :

Maximise profit $P = 22\,x + 18\,y$
subject to the constraints

$$x + y \leqslant 20$$
$$360\,x + 240\,y \leqslant 5760$$
$$x \geqslant 0\,,\ y \geqslant 0$$

## 1.4 Graphical Method of Solving Linear Programming Problem

Let us consider one linear programming problem and study how to solve it graphically.

**Example 1 :** Vikram has two machines with which he can manufacture either bottles or tumblers. The first of the two machines has to be used for one minute and the

second for two minutes in order to manufacture a bottle and the two machines have to be used for one minute each to manufacture a tumbler. During an hour the two machines can be operated for at the most 50 and 54 minutes respectively. Assuming that he can sell as many bottles and tumblers as he can produce, find how many of bottles and tumblers he should manufacture so that his profit per hour is max.mum, being given that he gets a profit of ten paise per bottle and six paise per tumbler.

**Solution :** Let us suppose that Vikram manufactures $x$ bottles and $y$ tumblers per hour to get the maximum profit. Then the first machine will be operated for $(x + y)$ minutes and the second will be operated for $(2x + y)$ minutes. Then according to the given conditions of the problem we have

$$x + y \leqslant 50 \qquad \ldots(1)$$

and

$$2x + y \leqslant 54 \qquad \ldots(2)$$

Also, either none or some bottles and either none or some tumblers are manufactured. and therefore,

$$x \geqslant 0 \qquad \ldots(3)$$

and

$$y \geqslant 0 \qquad \ldots(4)$$

We are required to find such a solution of the system of inequations (1) to (4) that the profit "$P$" paise per hour is maximum. Also the total profit per hour will be given by

$$P = 10x + 6y \qquad \ldots(5)$$

Thus, we have to so programme the manufacture of bottles and tumblers that the linear function of the two variables $x$ and $y$ defined by $P$ is maximised subject to the linear constrains (1) to (4). We are, therefore, justified in terming the above problem as a linear programming problem.

Mathematically, the problem reduce to maximising

$$P = 10x + 6y$$

subject to the conditions

$$x + y \leqslant 50$$
$$2x + y \leqslant 54$$
$$x \geqslant 0$$
$$y \geqslant 0$$

Now, the graph of the system of inequations can be easily obtained and is as shown in figure 1.7. (As $x \geqslant 0$ and $y \geqslant 0$, the graph has been drawn in the first quadrant only). The points in the region bounded by the quadrilateral $OABC$, including the boundary constitute the graph of the system of inequations. And so, any possible solution of the problem must occur at some points on the graph.

Equation (5) represents a line and as $P$ changes, the line moves parallel to itself. This line will reach a point on the boundary of the region (the graph of the system of inequations), before it reaches a point in the interior of the region. For maximum value of $P$, it reaches the boundary from above and for minimum, it reaches the boundary from below. If it reaches two vertices of the boundary simultaneously, it will be parallel to the segment joining the two points and every point of the segment will give a maximum (or minimum) value of $P$.

However, we are interested in that solution which gives a maximum value of $P$. We assume the following well known theorem without proof.

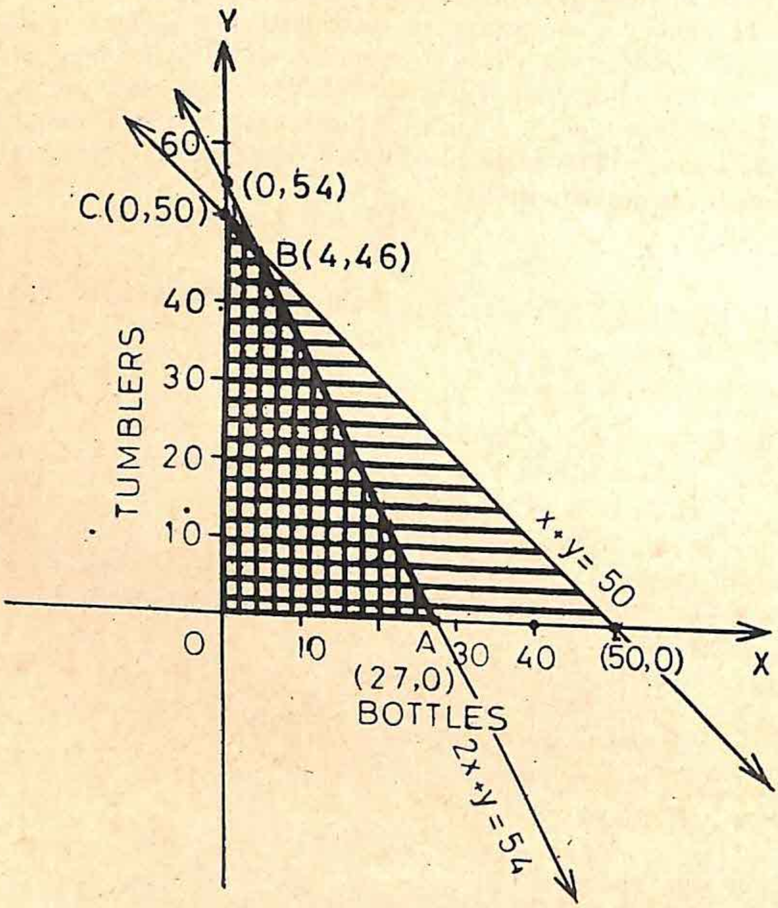



Fig. 1.7

**Theorem : Of all the points in a region bounded by a number of lines, if any point makes a linear expression P assume a maximum (or minimum) value, then it is so at a vertex point on the boundary of the region.**

In the present case the vertices of the bounded region, are $O$, $A$, $B$ and $C$. It is not difficult to see that the co-ordinates of these vertices are (0, 0), (27, 0), (4, 46) and (0, 50) respectively. Corresponding to these values of $x$ and $y$, $P$ takes the values

$$10\times0 + 6\times0,\ 10\times27 + 6\times0,\ 10\times4 + 6\times46,\ 10\times0 + 6\times50$$

*i.e.* 0, 270, 316, 300 paise

We see that the profit will be maximum provided he manufactures 4 bottles and 46 tumblers per hour and the profit per hour will be Rs 3.16.

**Example 2 :** Smita goes to the market to purchase buttons. She needs at least 20 large buttons and at least 30 small buttons. The shopkeeper sells buttons in two forms—(*i*) boxes and (*ii*) cards.

A box contains ten large and five small buttons and a card contains two large and five small buttons. Find the most economical way in which she should purchase the buttons, if a box costs 25 paise and a card ten paise only.

**Solution :** Let us suppose that she purchases $x$ cards and $y$ boxes of buttons. Then the total cost that she pays will be

$$10x + 25y \text{ paise}$$

We denote it by $C$, so that

$$C = 10x + 25y \qquad ...(1)$$

The number of large buttons she will get is

$2x + 10y$ and the number of small buttons is $5x + 5y$.

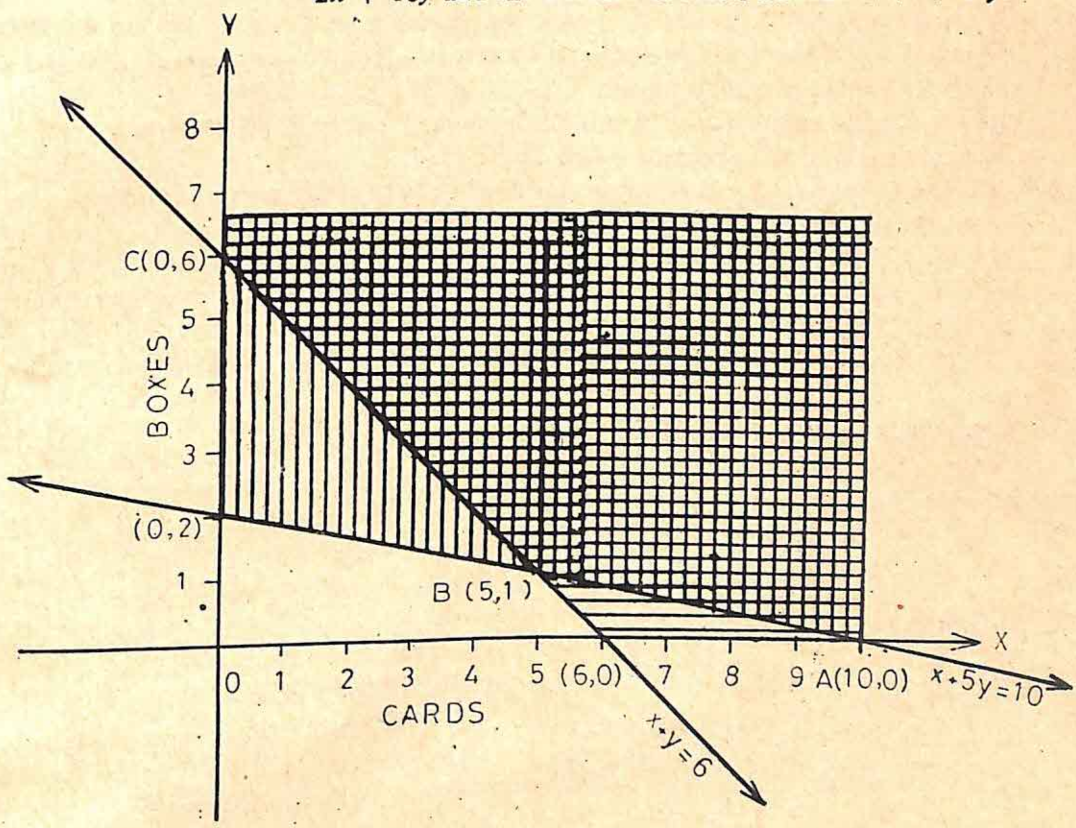



Fig. 1.8

As she must have 20 large and 30 small buttons, we get,

$$2x + 10y \geqslant 20 \text{ i.e } x + 5y \geqslant 10 \qquad ...(2)$$

$$5x + 5y \geqslant 30 \text{ i.e. } x + y \geqslant 6 \qquad ...(3)$$

Also either she has to purchase some cards or none and similar is the case for boxes

$\therefore$ $x \geqslant 0$ ...(4)

$y \geqslant 0$ ...(5)

The graph of the inequations (2) to (5) is shown in figure 1.8.

The vertices of the boundary of the graph are the points $A$ (10, 0), $B$ (5, 1) and $C$ (0, 6)
The corresponding cost will be 100, 50 + 25 and 150 paise. And so if she purchases 5 cards and one box, she has to pay the minimum *i.e.* 75 paise.

## Exercises 1.2

1. A furniture dealer deals in only two items *viz.* tables and chairs. He has Rs 5000 to invest and a space to store at most 60 pieces. A table costs him Rs 250 and a chair Rs 50. He can sell a table at a profit of Rs 50 and a chair at a profit of Rs 15. Assuming that he can sell all the items that he buys, prepare a mathematical model of the problem stated above.
2. A dietician wishes to mix two types of food in such a way that the vitamin contents of the mixture contains at least 8 units of vitamin $A$ and 10 units of vitamin $C$. Food $I$ contains 2 units per kg of vitamin A and 1 unit per kg of vitamin $C$, while the food $II$ contains 1 unit per kg of vitamin $A$ and 2 units per kg of vitamin $C$. It costs Rs 5 per kg to purchase food $I$ and Rs 7 per kg to purchase food $II$. Prepare a mathematical model of the problem stated above.
3. Find the maximum value of the function

$$C = 2x + 3y$$

subject to

$$\begin{aligned} x + 2y &\leqslant 10 \\ 2x + y &\leqslant 14 \\ x \geqslant 0, y &\geqslant 0 \end{aligned}$$

4. **Maximise**

$$C = 4x + 9y$$

Subject to

$$\begin{aligned} x + 5y &\leqslant 200 \\ 2x + 3y &\leqslant 134 \\ x &\geqslant 0 \\ y &\geqslant 0 \end{aligned}$$

5. **Minimise**

$$C = x + y$$

Subject to

$$3x + 2y \geqslant 12$$

$$x + 3y \geqslant 11$$
$$x \geqslant 0, y \geqslant 0$$

6. Sudesh wants to invest Rs 12000 in Saving Certificates and in National Saving Bonds. According to rules, she has to invest at least Rs 1000 in Saving Certificates and at least Rs 2000 in National Saving Bonds. If the rate of interest on Saving Certificates is 8% p.a. and the rate of 'nterest on the National Saving Bonds is 10% p.a., how should she invest her money to earn maximum yearly income? What is the maximum yearly income?

7. An aeroplane can carry a maximum of 200 passengers. A profit of Rs 400 is made on each first class ticket and a profit of Rs 300 is made on each economy class ticket. The airline reserves at least 20 seats for first class. However, at least 4 times as many passengers prefer to travel by economy class than by the first class. Determine how many of each type of tickets must be sold in order to maximise the profit for the airline. What is the maximum profit?

8. A small firm manufactures necklaces and bracelets. The combined number of necklaces and bracelets that it can handle per day is at most 24. The bracelet takes one hour to make and the necklace takes half an hour. The maximum number of hours available per day is 16. If the profit on the bracelet is Rs 2 and the profit on the necklace is Re 1, how many of each product should be produced daily to maximise profit?

9. A company manufactures two types of telephone sets, one of which is cordless. The cord type telephone set requires 2 hours to make, and the cordless model requires 4 hours. The company has at the most 800 work hours per day to manufacture these models and the packing department can pack at the most 300 telephone sets per day. If the company sells the cord type model for Rs 300 and the cordless model for Rs 400, how many of telephone sets of each type should it produce per day to maximise its sales?

10. A gardener has a supply of fertilizer of type I which consists of 10% nitrogen and 6% phosphoric acid and type II fertilizer which consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, he finds that he needs at least 14 kg of nitrogen and 14 kg of phosphoric acid for his crop. If the type I fertilizer costs 60 paise per kg and type II fertilizer costs 40 paise per kg, determine how many kilograms of each fertilizer should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?

CHAPTER 2

# Polynomials

## 2.1 Polynomials

We have already studied the concept of functions in class IX. In this chapter, we will study functions of a special type, called the polynomial functions. These functions are easier to work with, in comparison to many other functions occuring in mathematics and so we will study them in some detail.

Let us consider $p(x) = 3x^2 - 7x + 2$. We notice that for any real value of $x$, $p(x)$ is a real number. For example, $p(2) = 0$, $p(\frac{1}{2}) = -3/4$, $p(1) = -2$, etc. Thus, we have defined a function whose domain is the set of real numbers and whose range is a subset of the set of real numbers. A function of the type, as in example above, is called a polynomial function in $x$.

**Definition : A function $p(x)$ defined by $p(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$, where $a_0, a_1, \dots, a_n$ are real numbers and $n$ is a non-negative integer, is called a polynomial function in x, or briefly a polynomial in x, over reals.**

We shall take the domain of $p(x)$ to be the set of real numbers. Clearly its range is a subset of the set of real numbers.

Some examples of polynomials are

$$9x^3 - 4x^2 + 7x - 2,\ 5x^4 - \frac{8x^3}{3} + \frac{4x^2}{7} + 3x - \frac{5}{2},\ 5x + 8,$$
$$\frac{x^4}{4} - x^3 + 2\sqrt{2}\,x + 3, \text{ etc.}$$

A polynomial in $x$ over integers will mean a polynomial in which the coefficients of all the terms are integers and a polynomial over rationals will mean a polynomial in which the coefficients are all rationals. In this chapter, all the polynomials considered will be polynomials over reals.

We may emphasise that in a polynomial the exponent of $x$ in each term is a non-negative integer. Thus, whereas $2x + 3$ is a polynomial, $\frac{2}{x} + 3$ is not a polynomial, since the exponent of $x$ in the term $\frac{2}{x}$ is $-1$, which is not a non-negative integer.

*Footnote* : If in a polynomial, all the coefficients are zero, then it is called the zero polynomial.

Two polynomials are equal if and only if the coefficients of all like powers of $x$ in the two polynomials are equal. Thus, we may notice that the two polynomials $3x^2 + 5x - 8$ and $-8 + 3x^2 + 5x$ are equal; however, the two polynomials $5x^3 + 6x^2 + 2x - 8$ and $5x^3 - 2x^2 + 2x - 8$ are not equal since the coefficients of $x^2$ in the two are different.

We can always write a polynomial in such a way that the powers of $x$ are either in increasing or in decreasing order. These two ways of writing a polynomial are called *standard forms* of writing a polynomial. However, in this book, we will stick to writing a polynomial in the decreasing order of powers of $x$.

**Example 1 :** Write $6x^3 - 7x + 8 - 4x^4$ in the standard form.

**Solution :** The polynomial in the standard form is :

$$-4x^4 + 6x^3 - 7x + 8$$

**Example 2 :** Write the polynomial $-2x + x^3 + \frac{x^4}{3} + 7$ in the standard form.

**Solution :** The standard form of writing the polynomial is

$$\frac{x^4}{3} + x^3 - 2x + 7$$

## SOME SPECIAL TYPES OF POLYNOMIALS

Some polynomials have special names. After combining like powers of $x$ if we find that a polynomial consists of a single term, it is called a monomial. Examples of a monomial are 3, $8x$, $7x^3$, $2x + 7x$, etc. If the polynomial consists of two terms, it is called a binomial. Examples of a binomial are $2x + 5$, $x^2 - 7$, $3x^3 - 7$, $ax^2 + bx$ (where $a \neq 0$, $b \neq 0$), $3x + 4x^2 + x^2$, etc. If the polynomial consists of three terms, it is called a trinomial. For example $5x^3 + 8x - 7$, $3x^4 - x^2 + 2$, $x^2 + x + 1$ are trinomials.

## Exercises 2.1

1. Point out, which of the following functions are polynomials. Justify your answer.

(*i*) $x^2 + x + 2$ (*ii*) $2\sqrt{x} + 3$

(*iii*) $2x^2 + 5$ (*iv*) 8

(*v*) $\frac{\sqrt{3}}{x^2} + 8$ (*vi*) $\sqrt{3}\,x^2 - 5x$

2. Write the following polynomials in the standard form :

(*i*) $4x + x^3 + x^6 - 7x^2$ (*ii*) $5x^3 + 2x - x^2 - x^5$

(*iii*) $y^2 - 6y^3 + y^4 - 8y + 7$ (*iv*) $y^2 + y^4 + y^6 - y - y^3 - y^5$

(*v*) $\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4}$

3. In each of the following parts, point out which polynomials are over integers and justify your answer:

(i) $3x^4 - 4x^3 + 2x - 7$ (ii) $3x^4 - \frac{4}{3}x^3 + 2x - 7$

(iii) $\frac{2}{5}x^3 + \frac{1}{2}x^2 - \frac{4x}{5} + \frac{1}{6}$ (iv) $5x^5 + 4x^4 + 3x^3$

(v) $-x^3 - x^2 - x - 1$

4. In each of the following, write which polynomials are monomials, binomials or trinomials. Justify your answer.

(i) $7x^2 + 2x$ (ii) $x^3 + 1$

(iii) $x^2 + x + 1$ (iv) $x^2$

(v) $7$ (vi) $4x^2 + 4x$

(vii) $2x^3$ (viii) $5x^3 + 8x^3$

(ix) $x^3 + x - 7$

## 2.2 Sum, Difference and Product of two Polynomials, Degree of a Polynomial

Let $p(x) = 3x^3 + 5x - 4$ and $q(x) = 6x^4 - x + 2$ be two polynomials. We have the sum of $p(x)$ and $q(x)$, written as $p(x) + q(x)$, to be $(3x^3 + 5x - 4) + (6x^4 - x + 2)$ $= 6x^4 + 3x^3 + 4x - 2$. Thus the sum of two polynomials is obtained by adding the terms containing like powers of $x$, occuring in the two polynomials.

Similarly, $p(x) - q(x) = (3x^3 + 5x - 4) - (6x^4 - x + 2)$
$= -6x^4 + 3x^3 + 6x - 6$

We notice that $p(x) - q(x)$ is obtained by subtracting terms occuring in $q(x)$ from the terms occuring in $p(x)$ and arranging the resulting polynomial in the standard form.

Let us give some examples to find the product of two polynomials.

Let $p(x) = 3x$ and $q(x) = 5x^2 + x + 6$

Then
$$\begin{aligned} p(x)\,q(x) &= 3x\,(5x^2 + x + 6) \\ &= 3x \cdot 5x^2 + 3x \cdot x + 3x \cdot 6 \\ &= 15x^3 + 3x^2 + 18x \end{aligned}$$

Let us find the product of two polynomials in another example when $p(y) = 3y + 2$ and $q(y) = y^2 - y - 6$.

Then
$$\begin{aligned} p(y)\,q(y) &= (3y + 2)(y^2 - y - 6) \\ &= 3y\,(y^2 - y - 6) + 2\,(y^2 - y - 6) \\ &= 3y^3 - 3y^2 - 18y + 2y^2 - 2y - 12 \\ &= 3y^3 - y^2 - 20y - 12 \end{aligned}$$

Thus, the product of two polynomials in $x$ is obtained by using the distributive law repeatedly.

We now define the degree of a polynomial. **By the degree of a polynomial in x we mean the exponent of the term containing the highest power of x.** Clearly it is a non-negative integer. Thus, the degree of $5x^3 - 3x^2 + 2x + 1$ is 3, the degree of $-6x + 5$ is 1 and the degree of 4 is zero. We illustrate the above ideas with a few examples.

**Example 1 :** Find the sum of the two polynomials
$$4x^2 - 3x + 2 \text{ and } 3x^4 + 2x^3 + 6x - 7$$

**Solution :** The sum of the two polynomials is
$$(4x^2 - 3x + 2) + (3x^4 + 2x^3 + 6x - 7)$$
$$= 3x^4 + 2x^3 + 4x^2 + 3x - 5$$

**Example 2 :** Find the sum of the two polynomials
$$3y^5 - 4y^3 + y^2 + y - 7 \text{ and } - 3y^5 + 4y^3 + 6y^2 + 7y + 2$$

**Solution :** The required sum is
$$(3y^5 - 4y^3 + y^2 + y - 7) + (- 3y^5 + 4y^3 + 6y^2 + 7y + 2)$$
$$= 7y^2 + 8y - 5$$

**Example 3 :** Let $p(y) = 6y^2 + y - 2$ and $q(y) = - 8y^3 + 3y^2 + 7$.
Find $p(y) - q(y)$.

**Solution :** $p(y) - q(y) = (6y^2 + y - 2) - (- 8y^3 + 3y^2 + 7)$
$$= 8y^3 + 3y^2 + y - 9$$

**Example 4 :** If $p(x) = 8x^3 + 6x^2 + 5x + 2$ and $q(x) = 8x^3 - 7x^2 + 4x + 2$, what is $p(x) - q(x)$ ?

**Solution :** $p(x) - q(x) = (8x^3 + 6x^2 + 5x + 2) - (8x^3 - 7x^2 + 4x + 2)$
$$= 13x^2 + x$$

**Example 5 :** Find the product of the polynomials
$$4x^2 + x - 2 \text{ and } 3x^2 + 6x + 4$$

**Solution :** $(4x^2 + x - 2)(3x^2 + 6x + 4)$
$$= 4x^2(3x^2 + 6x + 4) + x(3x^2 + 6x + 4) - 2(3x^2 + 6x + 4)$$
$$= 12x^4 + 24x^3 + 16x^2 + 3x^3 + 6x^2 + 4x - 6x^2 - 12x - 8$$
$$= 12x^4 + 27x^3 + 16x^2 - 8x - 8$$

**Example 6 :** Find the product of the polynomials $p(x)$ and $q(x)$ where $p(x) = 0$ and $q(x) = x^2 + 5$

**Solution :** $p(x)\,q(x) = 0 \cdot (x^2 + 5) = 0$

We have already learnt addition, subtraction and multiplication of two polynomials. Also we know the meaning of degree of a polynomial. We now will try to see whether there is any relationship between the degrees of two polynomials and their sum, difference and product.

If we have a look at examples 1 and 2 above we see that the degree of the sum of two polynomials is equal to or less than the degree of the polynomial of higher degree. Similarly examples 3 and 4 above show that the degree of the difference of two polynomials is equal to or less than the degree of the polynomial of higher degree. About the degree of the product of two polynomials, examples 5 and 6 help us. We find that if none of the two polynomials is zero then the degree of the product is equal to the sum of the degrees of the two polynomial factors. In case when one polynomial is zero then the product is zero (zero polynomial) and hence the degree of the product in this case is zero.

We may state this now as a general rule :

**The degree of the sum or difference of two polynomials is less than or equal to the degree of the polynomial of higher degree, and the degree of the product of two non-zero polynomials is equal to the sum of the degrees of the two polynomials.**

Till now we have been dealing with a polynomial in one variable. We now give below some examples of sum, difference and product of polynomials in two variables $x$ and $y$. A polynomial in two variables $x$ and $y$ is usually denoted by $p(x, y)$, $q(x, y)$, etc.

**Example 7 :** Find the sum of the polynomials

$$x^3 + 3xy^2 + y^3 \text{ and } 3xy - y^2 + y$$

**Solution :** The sum is

$$(x^3 + 3xy^2 + y^3) + (3xy - y^2 + y)$$
$$= x^3 + 3xy^2 + y^3 + 3xy - y^2 + y$$

**Example 8 :** If $p(x, y) = x^2 + y^2 + 2xy$ and $q(x, y) = x^2 + y^2 - 3xy$, then find $3p(x, y) - 4q(x, y)$

**Solution :** $3p(x, y) - 4q(x, y) = 3(x^2 + y^2 + 2xy) - 4(x^2 + y^2 - 3xy)$

$$= (3x^2 + 3y^2 + 6xy) - (4x^2 + 4y^2 - 12xy)$$
$$= - x^2 - y^2 + 18xy$$

**Example 9 :** Find the product of $x^2 + xy + y^2$ and $x - y$

**Solution :** The required product is

$$(x^2 + xy + y^2)(x - y)$$
$$= x^2(x - y) + xy(x - y) + y^2(x - y)$$
$$= x^3 - x^2y + x^2y - xy^2 + y^2x - y^3$$
$$= x^3 - y^3$$

In the same manner, we can find the sum, difference and product of polynomials in more than two variables.

**The following formulae are important, as they will be used many times in solving problems. You may verify these by actual multiplication.**

(i) $(x + y)^2 = x^2 + 2xy + y^2$

(ii) $(x - y)^2 = x^2 - 2xy + y^2$

(iii) $(x + y)(x - y) = x^2 - y^2$

(iv) $(x + y)^3 = x^3 + 3xy(x + y) + y^3$

(v) $(x - y)^3 = x^3 - 3xy(x - y) - y^3$

(vi) $(x+y+z)^2 = x^2+y^2+z^2 + 2xy + 2yz + 2zx$

## Exercises 2.2

1. Find the sum of the polynomials in each of the following parts and find the degree of the sum ;

   (*i*) $x^2 - 3x + 5;\ 3x^2 + 3x - 10$

(*ii*) $3x^3 + x^2 - x - 8; -3x^3 + 5x + 6$
(*iii*) $y^3 + 3y^2 + 3y + 1; y^3 - 3y^2 + 3y - 1$
(*iv*) $z^4 + 6z^3 + 10z^2 + 6z + 1; -z^4 + 6z^3 - 10z^2 + 6z - 1$
(*v*) $z^2 + z - 7; z^3 + z^2 + 3z + 6$
(*vi*) $y^2 - 3y + 5; -3y^2 + 7y + 5; y^2 + 7$

2. In the following, subtract the second polynomial from the first and find the degree of the difference:
   (*i*) $y^3 + 3y^2 + 3y + 1; y^3 - 3y^2 + 3y - 1$
   (*ii*) $x^4 + 2x^2 + 1; x^4 - 2x^2 + 1$
   (*iii*) $x^3 + 3x^2 + x + 1; x^4 + 5x^3 + 6x^2 + 5x + 2$
   (*iv*) $y^4 - 4y^3 + 6y^2 - 4y + 1; y^2 + 2y + 1$
3. Subtract $6x^2 + 12x + 5$ from the sum of
   $x^3 + 4x^2 + 6x + 1$ and $2x^3 + 6x + 7$
4. What should be subtracted from the sum of
   $y^3 + 2y + 1$ and $y^3 + 6y - 2$ to get 1 ?
5. Find the product of the polynomials in each part and also find the degree of the product :
   (*i*) $x + 4; x + 6$
   (*ii*) $y + 2z; y + 3z$
   (*iii*) $x^2 - 2x + 1; -1 + x$
   (*iv*) $x^2 + 5x + 6; x + 1$
   (*v*) $x^3 + x + 1; x - 1$
   (*vi*) $x^2 + 2xy + y^2; x^2 + y^2$
   (*vii*) $x^2 + x + 1; x^2 - x + 1$
   (*viii*) $x^2 + y^2 + z^2 - xy - yz - zx; x + y + z$
6. Simplify the following :
   (*i*) $(x - 1)(x + 1)(x^2 + 1)$
   (*ii*) $(y + 1)(y + 3)(y + 5)$

### 2.3 Division of two Polynomials

We are familiar with division of one number by another non-zero number. When we divide one number by another, we get one number as quotient and another number as remainder, which is zero or a number less than the divisor. The remainder is zero when the divisor is a factor of the dividend.

A similar situation arises when we divide a polynomial $f(x)$ by a polynomial $g(x)$. Just as in the case of numbers, here also the divisor $g(x) \neq 0$. We shall write the quotient and the remainder as $q(x)$ and $r(x)$ respectively.

We shall solve now a few examples on division. Before we start division process, we must write the two polynomials in the standard form.

**Example 1 :** Divide $f(x)$ by $g(x)$ where
$f(x) = x^3 - 6x^2 + 11x - 6$ and $g(x) = x^2 - 5x + 6$

**Solution :**

$$
\begin{array}{r|l}
 & x-1 \\
\hline
x^2-5x+6 & x^3-6x^2+11x-6 \\
 & x^3-5x^2+6x \\
\hline
 & -x^2+5x-6 \\
 & -x^2+5x-6 \\
\hline
 & 0
\end{array}
$$

Here $q(x) = x - 1$ and $r(x) = 0$

**Notice that the process stops when the degree of the remainder is less than the degree of the divisor.**

*Note* : **As in this case, when $r(x) = 0$, we say that the divisor $g(x)$ is a factor of the dividend $f(x)$.**

**Example 2 :** Divide $x^3-14x^2+37x-60$ by $x-2$

**Solution :**

$$
\begin{array}{r|l}
 & x^2-12x+13 \\
\hline
x-2 & x^3-14x^2+37x-60 \\
 & x^3-2x^2 \\
\hline
 & -12x^2+37x-60 \\
 & -12x^2+24x \\
\hline
 & 13x-60 \\
 & 13x-26 \\
\hline
 & -34
\end{array}
$$

**In** this example $q(x)=x^2-12x+13$ and $r(x)=-34$. Notice that here the divisor is not a factor of the dividend, since $r(x) \neq 0$.

**Example 3 :** Divide $f(x)=5x^3-70x^2+153x-342$ by $g(x)=x^2-10x+16$

**Solution :**

$$
\begin{array}{r|l}
 & 5x-20 \\
\hline
x^2-10x+16 & 5x^3-70x^2+153x-342 \\
 & 5x^3-50x^2+80x \\
\hline
 & -20x^2+73x-342 \\
 & -20x^2+200x-320 \\
\hline
 & -127x-22
\end{array}
$$

Here $q(x)=5x-20$ and $r(x)=-127x-22$.

**Example 4 :** Let $f(x)=x^2+2$ and $g(x)=x^3+7x$. Divide $f(x)$ by $g(x)$. Find the quotient and the remainder.

**Solution :** In this example since the degree of the dividend is less than the degree of the divisor, the quotient $q(x)$ is 0 and the remainder $r(x)=x^2+2$.

In each of the examples above we notice that the degree of the remainder is less than he degree of the divisor. There is a famous result, known as *division algorithm*, which state as follows.

**Division algorithm :** Let $f(x)$ and $g(x)$ be two polynomials and $g(x) \neq 0$. Then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x)=q(x)\, g(x)+r(x)$$

where degree of $r(x)$ is less than the degree of $g(x)$. (The proof of this theorem is beyond the scope of the book.)

## Exercises 2.3

In each of the following questions $f(x)$ and $g(x)$ are given. Find the quotient $q(x)$ and the remainder $r(x)$ when $f(x)$ is divided by $g(x)$.

1. $f(x)=2x^3-11x^2+19x-10,\ g(x)=2x-5$
2. $f(x)=x^3-3x^2-x+3,\ g(x)=x^2-4x+3$
3. $f(x)=x^4+2x^3+3x^2+2x+20,\ g(x)=x^2+2x+2$
4. $f(x)=x^3+6x^2+10x+16,\ g(x)=x+3$
5. $f(x)=x^4+1,\ g(x)=x-1$
6. $f(x)=x^6+5x^3+7x+3,\ g(x)=x^2+2$
7. $f(x)=x^4+2x^2+1,\ g(x)=x^3+1$
8. $f(x)=x^3+x^2-x-1,\ g(x)=x^4+1$
9. $f(x)=x^6+3x^2+10,\ g(x)=x+3$
10. $f(x)=3x^2+x+1,\ g(x)=4x$

### 2.4 Remainder Theorem

We have learnt above, the division of one polynomial by another non-zero polynomial. We are now interested in finding out as to what happens if the divisor is a polynomial of degree 1. By division algorithm we see that in this case remainder $r(x)$ is of degree zero and hence is a real number. We will try to find the remainder in the following two examples.

**Example 1 :** Let $f(x)=x^3+2x^2-7x+3$ and $g(x)=x-4$. Find the remainder when $f(x)$ is divided by $g(x)$.

**Solution :**

$$\begin{array}{r|l} & x^2+6x+17 \\ \hline x-4 & x^3+2x^2-7x+3 \\ & x^3-4x^2 \\ \hline & \quad 6x^2-7x+3 \\ & \quad 6x^2-24x \\ \hline & \qquad 17x+3 \\ & \qquad 17x-68 \\ \hline & \qquad\quad 71 \end{array}$$

The remainder is 71. We may check that $f(4)$ is also 71.

**Example 2** : Find the remainder when $x^2+5x+4$ is divided by $x-a$.

**Solution :**

$$
\begin{array}{r|l}
x-a & \overline{x^2+5x+4} \quad x+(a+5) \\
 & \underline{x^2-ax} \\
 & (a+5)x+4 \\
 & \underline{(a+5)x-a(a+5)} \\
 & a(a+5)+4
\end{array}
$$

This time the remainder is $a(a+5)+4=a^2+5a+4$. If the dividend is denoted by $f(x)$ then we see that the remainder is $f(a)$.

The result is general and we state and prove it below as a theorem.

**Remainder Theorem**

*If a polynomial $f(x)$ is divided by $x-a$ then the remainder is $f(a)$, where $a$ is a real number.*

**Proof :** Let $f(x)$ be divided by $x-a$, and let the remainder be a real number, say $r$. Let the quotient be $q(x)$. Then by division algorithm we get

$$f(x)=(x-a)\cdot q(x)+r$$

Putting $x=a$, we get $f(a)=0\cdot q(x)+r$

*i.e.,* $\quad f(a)=r$

Thus, the remainder $r=f(a)$

This theorem helps us in finding the remainder, without actual division when the divisor is of the form $x-a$.

We now take two examples of the application of remainder theorem.

**Example 1 :** $f(x)=x^6-3x^5+2x^2+8$. Without actual division find the remainder when $f(x)$ is divided by $x-3$.

**Solution :** Using the remainder theorem we see that the remainder is $f(3)$.

Now, $f(3)=3^6-3\cdot 3^5+2\cdot 3^2+8$

$=26$

Thus, the remainder is 26.

**Example 2 :** Without using division process, find the remainder when $x^3+4x^2+6x-2$ is divided by $x+5$.

**Solution :** The divisor here is $x+5=x-(-5)$.

Hence the remainder, on division by $x+5$, is $f(-5)$.

But $f(-5)=(-5)^3+4(-5)^2+6(-5)-2=-57$

Hence the remainder is $-57$.

## Exercises 2.4

1. $f(x)=x^3-2x^2+x-2$. Find the remainder, by using division process and also by using remainder theorem when $f(x)$ is divided by
   (i) $x-1$ (ii) $x-2$ (iii) $x+1$
2. $f(x)=x^3-6x^2+11x-6$. Find the remainder when the divisor is
   (i) $x-1$ (ii) $x-4$ (iii) $x+2$
3. $f(x)$ is the polynomial $4x^3-12x^2+11x-3$. Use the remainder theorem to find the remainder when $f(x)$ is divided by
   (i) $x-\dfrac{1}{2}$ (ii) $x+\dfrac{1}{2}$ (iii) $x-\dfrac{3}{2}$
4. In each of the following parts show that $g(y)$ is a factor of $f(y)$:
   (i) $f(y)=y^3-8$ and $g(y)=y-2$
   (ii) $f(y)=y^3+4y^2+5y+2$ and $g(y)=y+1$
   (iii) $f(y)=y^4-y^2-12$ and $g(y)=y+2$

**Hint :** In each of the above, show that the remainder is zero when $f(y)$ is divided by $g(y)$.

### 2.5 Factor Theorem

So far, we have gained some information which we may use to factorise polynomials in some cases. Now we prove a theorem called Factor Theorem. It tells us precisely when a linear polynomial of the form $(x-a)$ is a factor of the given polynomial.

**Factor Theorem :** *Let $f(x)$ be a polynomial and 'a' be a real number. Then the following two results hold :*

(i) *If $f(a)=0$ then $(x-a)$ is a factor of $f(x)$.*

(ii) *If $(x-a)$ is a factor of $f(x)$ then $f(a)=0$.*

**Proof :** By division algorithm we have

$$f(x)=(x-a)\,q(x)+r \quad \text{...(1)}$$

By remainder theorem we have $r=f(a)$ ...(2)

Using (1) and (2) we have

$$f(x)=(x-a)\,q(x)+f(a) \quad \text{...(3)}$$

To prove (i) : Let $f(a)=0$ then from (3) we have $f(x)=(x-a)\,q(x)$

Hence $(x-a)$ is a factor of $f(x)$.

To prove (ii) : Let $(x-a)$ be a factor of $f(x)$ so the remainder is 0 when we divide $f(x)$ by $x-a$.

But the remainder is $f(a)$

So $f(a)=0$.

We shall now illustrate the use of this theorem in the following examples:

**Example 1 :** Let $f(x)=x^3-12x^2+44x-48$. Find out whether $x-2$ and $x-3$ are factors of $f(x)$.

**Solution :** To check whether $x-2$ is a factor of $f(x)$ we find $f(2)$.

Now $f(2)=2^3-12\cdot 2^2+44\cdot 2-48=0$

Hence by factor theorem, $(x-2)$ is a factor of $f(x)$.

Also $f(3)=3^3-12\cdot 3^2+44\cdot 3-48=3$

Since $f(3)\neq 0$, $(x-3)$ is not a factor of $f(x)$.

**Example 2 :** Let $f(x)=x^3-kx^2+11x-6$. For what value of $k$, is $(x-1)$ a factor of $f(x)$?

**Solution :** Since $(x-1)$ is a factor of $f(x)$,

$f(1)=0$. (by factor theorem)

But $f(1)=1^3-k\cdot 1^2+11\cdot 1-6=6-k$

$\therefore \quad 6-k=0$

Hence $k=6$

**Example 3 :** Let $f(x)=x^3+kx^2+hx+6$. Find the values of $h$ and $k$ so that $(x+1)$ and $(x-2)$ are factors of $f(x)$.

**Solution :** Since $(x+1)$ is a factor of $f(x)$ so $f(-1)=0$ (by factor theorem)

But $f(-1)=(-1)^3+k(-1)^2+h(-1)+6$

$=k-h+5$

$\therefore \quad k-h+5=0 \quad \ldots(1)$

Since $(x-2)$ is a factor of $f(x)$, $f(2)=0$ and we have, in the same way as above,

$8+4k+2h+6=0$

$\Leftrightarrow \quad 4k+2h+14=0$

$\Leftrightarrow \quad 2k+h+7=0 \quad \ldots(2)$

Solving (1) and (2) we get $k=-4$ and $h=1$.

## Exercises 2.5

In the problems 1–5, use the factor theorem to check whether or not the polynomial $g(x)$ is a factor of the polynomial $f(x)$.

1. $f(x)=x^4+x^3-x^2-x-18$ ; $g(x)=x-2$
2. $f(x)=x^5+3x^4+x^2+8x+15$ ; $g(x)=x+3$
3. $f(x)=x^3+x^2+x+1$ ; $g(x)=x-1$
4. $f(x)=x^4+4x^2+x+6$ ; $g(x)=x+2$
5. $f(x)=x^3+x^2+3x-4$ ; $g(x)=x+5$
6. If $(x-1)$ is a factor of $a^2x^2-3ax+3a-1$, find the values of $a$.
7. Find the values of $p$ and $q$ so that $(x+2)$ and $(x-1)$ are factors of the polynomial $x^3+10x^2+px+q$.

## 2.6 Zeros of a Polynomial

We have studied polynomial functions in some detail. In this section we will be interested in studying zeros of a polynomial. *By the zero of a polynomial $f(x)$ we mean a real number $a$ such that $f(a)=0$.*

We illustrate the meaning of zeros of a polynomial through two examples.

**Example 1 :** Show that 1 is a zero of the polynomial

$$x^3-6x^2+11x-6.$$

**Solution :** Let $f(x)=x^3-6x^2+11x-6$

Now we find $f(1)$

$$f(1)=1^3-6\cdot 1^2+11\cdot 1-6=0$$

Since $f(1)$ is zero, 1 is a zero of the polynomial $f(x)$.

**Example 2 :** Show that 2 is not a zero of the polynomial

$$f(y)=y^3-y^2-y+1$$

**Solution :** Let us calculate $f(2)$

$$f(2)=2^3-2^2-2+1=3$$

Since $f(2)\neq 0$ so 2 is not a zero of the polynomial

$$y^3-y^2-y+1.$$

It is possible that sometimes a polynomial has no zero. Let us consider an example.

**Example 3 :** Show that $x^2+4x+5$ has no zero.

**Solution :** Let $f(x)=x^2+4x+5$

Then* $f(x)=(x^2+4x+4)+1$

$$=(x+2)^2+1$$

We notice that for all real values of $x$, $(x+2)^2$ is always non-negative. Hence $f(x)$ has always a value greater than or equal to 1. Thus, $f(x)$ has no zero.

If we are given a polynomial $f(x)$, we can test whether any given real number $a$ is a zero of $f(x)$ or not. This is what has been done in examples 1 and 2 in this section.

However, if we are given a polynomial $f(x)$, then finding its zeros is a more difficult task.

Sometimes zeros may not exist, as in example 3 above.

We will now state a theorem and use it to find the zeros of some polynomials. The theorem may be applied to polynomials of any degree. But it tells us only about integral zeros.

**Theorem :** Let $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n$ be a polynomial with integral co-efficients. If an integer $k$ is a zero of the polynomial, then $k$ is a factor of $a_n$. (The proof of the theorem is beyond the scope of the book.) We shall illustrate the usefulness of the theorem in the following examples:

**Example 4 :** Find all the integral zeros of the polynomial

$$f(y)=y^3-2y^2+y+4$$

**Solution :** Suppose $k$ is an integral zero of the polynomial $f(y)$. Then, by the above theorem, $k$ is a factor of 4. Hence possible values of $k$ are

$$1,\ -1,\ 2,\ -2,\ 4 \text{ and } -4.$$

*We break up the constant term so that the terms containing $x$ form a complete square.

We will now test each one of them to see whether it is a zero of the polynomial.

$f(1)=1^3-2\cdot1^2+1+4=4$ ; since $f(1)\neq0$, so 1 is not a zero of $f(y)$

$f(-1)=(-1)^3-2(-1)^2+(-1)+4=0$, so $-1$ is a zero of $f(y)$

We may now check for 2, $-2$, 4 and $-4$ in the same manner. We will find that they are not zeros of the polynomial $f(y)$. Thus, the only integral zero of $f(y)$ is $-1$. If there are other zeros then they are not integers.

**Example 5** : Find the integral zeros of the polynomial

$f(x)=12x^3-4x^2-3x+1$

**Solution** : Suppose that $k$ is a zero of the polynomial. Then $k$ is a factor of 1. Hence the only possible values of $k$ are 1 and $-1$.

We will now check whether $+1$ and $-1$ are zeros of the polynomial.

Now $f(1)=12\cdot1^3-4\cdot1^2-3\cdot1+1=6$

Hence 1 is not a zero of $f(x)$

Also $f(-1)=12(-1)^3-4(-1)^2-3(-1)+1=-12$

Hence $-1$ is also not a zero of $f(x)$.

Thus, the polynomial has no integral zeros.

*Remark* : In the above example, we may verify that $\frac{1}{2}$ is a zero of the polynomial. Thus if a polynomial has no integral zero, it does not mean that it has no zeros.

The general methods of finding zeros of a polynomial of degree 3 or degree 4 are not to be studied in this book. However, in special cases, we may be able to find all the zeros of such polynomials. The zeros of a polynomial of degree 2 will be studied in full detail in the next chapter.

## Exercises 2.6

1. Show that 1, 2 and 3 are zeros of the polynomial $y^3-6y^2+11y-6$.
2. Show that the zeros of the polynomial $y^3+12y^2+41y+30$ are $-1$, $-5$ and $-6$.
3. Show that $\frac{1}{2}$ is a zero of the polynomial

   $2x^2+7x-4$
4. Show that 2 is not a zero of the polynomial

   $x^2-7x+12$
5. Which of the numbers 2, 3 and $-4$ are the zeros of the polynomial

   $2x^2+7x-4$ ?

6. Show that the following polynomials have no zeros :
   (*i*) $x^2+4x+7$
   (*ii*) $4x^2+12x+11$
   (*iii*) $z^4+4z^2+5$
7. Find the integral zeros of the following polynomials :
   (*i*) $6x^3+23x^2-5x-4$
   (*ii*) $4y^3-8y^2-y+2$
   (*iii*) $z^4+4z^3+2z^2-4z-3$
8. Prove that the following polynomials have no integral zeros :
   (*i*) $z^3+4z+7$
   (*ii*) $2z^4+4z^3+3z^2-2z-2$

CHAPTER 3

# Quadratic Equations and Their Applications

### 3.1 Linear Factors of a Quadratic Polynomial

In the previous chapter we have learnt how to factorise the polynomials of ai degree by remainder theorem. However, in this section we will limit ourselves factorisation of the quadratic polynomial into linear factors.

Is the product of two linear polynomials a quadratic polynomial ?

If we multiply any two linear polynomials such as $3x+4$ and $2x+1$, then the produ is a quadratic polynomial which is $6x^2+11x+4$ in this case.

In general, if we multiply two linear polynomials $lx+m$ and $px+q$, where $l\neq$ $p\neq 0$ then the product of these polynomials is the quadratic polynomial $lpx^2+(lq+mp)x+mq$ which, of course, is of the standard form $ax^2+bx+c$ where $a=l$ $b=lq+mp$ and $c=mq$. Thus, we say that $lx+m$ and $px+q$ are the factors of t quadratic polynomial $lpx^2+(lq+mp)x+mq$. Hence, we find that the quadrat polynomial $ax^2+bx+c$ can be factorised, if there exist two numbers $r(=lq)$ and $s(=m$ such that $r+s(=lq+mp)=b=$co-efficient of $x$

and $rs(=lq\cdot mp=lp\cdot mq)=ac$

$=$(coefficient of $x^2$)$\times$(constant term)

Once we find two numbers with the above two properties, we break up the middle ter $bx$ of the quadratic polynomial into the SUM OR DIFFERENCE of two terms ar factorise the polynomial by grouping the terms and finding common factors. In ord to understand the method in more details, we consider the following examples:

**Example 1 :** Factorise $2x^2+11x+5$

**Solution :** Here $a=2$, $b=11$ and $c=5$

So we have to find two numbers $r$ and $s$ such that

$$r+s=b=11 \text{ and } rs=ac=2\times 5=10$$

Now what are the two integers whose sum is 11 and whose product is 10 ?

We note that the product of $r$ and $s$ is positive and also the sum of $r$ and $s$ is positiv Hence $r$ and $s$ are positive. So we should determine the positive integers whose sum 11 and product is 10. These numbers are 10 and 1, since $10\times 1=10$ ; $10+1=11$. No we break up the middle term $11x$ of the polynomial into sum of the two terms a $1\cdot x+10x$ in the given polynomial.

$$2x^2+11x+5=2x^2+10x+1\cdot x+5$$

$$=2x(x+5)+1(x+5)$$
$$=(x+5)(2x+1)$$

**Example 2 :** Factorise $3x^2-17x+10$

**Solution :** Here $a=3$, $b=-17$ and $c=10$

So find $r$ and $s$ such that

$r+s=b=-17$ and $rs=ac=3\times 10=30$

Since the product of $r$ and $s$ is positive, either $r$ and $s$ will both be positive or both negative. But the sum of $r$ and $s$ is negative. *So it is necessary to find $r$ and $s$ such that both the numbers are negative. Thus, the numbers are $-15$ and $-2$ since $(-15)(-2)=30$, $-15+(-2)=-17$.*

We break up the middle term of the given polynomial into $(-15x)+(-2x)$

$$\therefore\quad 3x^2-17x+10=3x^2-15x-2x+10$$
$$=3x(x-5)-2(x-5)$$
$$=(x-5)(3x-2)$$

**Example 3 :** Resolve into factors $14x^2+29x-15$

**Solution :** Here $a=14$, $b=29$ and $c=-15$

$\therefore\quad r+s=b=29$, $rs=ac=14\times(-15)=-210$

Since the product of $r$ and $s$ is negative, either $r$ or $s$ will be negative and the other will be positive. But since the sum of $r$ and $s$ is positive the numerically greater factor of $-210$ will be given positive sign and the other factor will be given negative sign.

Obviously, the required numbers are 35 and $-6$ since

$$35(-6)=-210 \text{ and } 35-6=29$$

The middle term $29x$ of the polynomial must be written as $35x-6x$.

$$\therefore\quad 14x^2+29x-15=14x^2+35x-6x-15$$
$$=7x(2x+5)-3(2x+5)$$
$$=(2x+5)(7x-3)$$

**Example 4 :** Factorise : $6x^2-5x-21$

**Solution :** Here $a=6$, $b=-5$ and $c=-21$

$\therefore\quad r+s=b=-5$ and $rs=ac=6(-21)=-126$

Since the product of $r$ and $s$ is negative and their sum is also negative the numerically greater of the two factors of $-126$ shall be given the negative sign and the smaller one the positive sign. We find that the required numbers are $-14$ and 9 since

$$-14\times 9=-126,\ -14+9=-5$$

Thus we break up the middle term $-5x$ of the given polynomial into the sum of the two terms as $-14x+9x$.

$$\therefore\quad 6x^2-5x-21=6x^2-14x+9x-21$$
$$=2x(3x-7)+3(3x-7)$$
$$=(3x-7)(2x+3)$$

## CONDITION FOR FACTORISATION OF A QUADRATIC POLYNOMIAL

In the previous part of this section we have studied that a quadratic polynomial is expressible as a product of two linear polynomials. Is it true that every quadratic polynomial is a product of two linear polynomials ? The answer to this question is not

in the affirmative in that it will be seen that not every quadratic polynomial is expressible as a product of two linear polynomials. For example it is not possible to express $x^2+1$ or $x^2+x+1$ as a product of two linear polynomials (with real co-efficients). Now we shall determine the condition under which the quadratic polynomial can be expressed as a product of linear polynomials.

Consider the polynomial $ax^2+bx+c$, $a\neq 0$

$$ax^2+bx+c=a\left[x^2+\frac{bx}{a}+\frac{c}{a}\right] \qquad ...(1)$$

$$=a\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right]\left[\text{add and subtract } \left(\frac{b}{2a}\right)^2\right]$$

$$=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right] \qquad ...(2)$$

Now three cases arise according as $b^2-4ac$ is

(*i*) positive (*ii*) zero (*iii*) negative

*Case I* : $b^2-4ac>0$

When $b^2-4ac>0$, $\sqrt{b^2-4ac}$ is defined.

Hence from (2), we have

$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right]$$

$$=a\left[\left(x+\frac{b}{2a}\right)-\frac{\sqrt{b^2-4ac}}{2a}\right]\cdot\left[\left(x+\frac{b}{2a}\right)+\frac{\sqrt{b^2-4ac}}{2a}\right]$$

$$=a\left[x+\frac{b-\sqrt{b^2-4ac}}{2a}\right]\cdot\left[x+\frac{b+\sqrt{b^2-4ac}}{2a}\right] \qquad ...(3)$$

So we have two linear factors which are different.

*Case II* : $b^2-4ac=0$

Now (2) reduces to

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2$$

$$=a\left(x+\frac{b}{2a}\right)\left(x+\frac{b}{2a}\right)$$

So the polynomial $ax^2+bx+c$ is expressible as the product of two coincident linear factors.

*Case III* : $b^2-4ac<0$

Let $r$ and $s$ be two numbers such that $r+s=b$ and $rs=ac$, then $b^2-4ac<0$ gives $(r+s)^2-4rs<0$ *i.e.*, $(r-s)^2<0$, which is not true. This shows that we cannot find $r$ and $s$ with the given properties. Hence we cannot express the given polynomial as a product of two linear factors.

**Example 5** : Factorise $3x^2+2x+1$

**Solution** : Here $a=3$, $b=2$ and $c=1$

In this case, $b^2-4ac=(2)^2-4\times 3\times 1$

$$=-8<0$$

Therefore, we cannot factorise the given polynomial.

**Example 6 :** Factorise : $2x^2+3x-7$

**Solution :** Here we have $a=2$, $b=3$ and $c=-7$

Now we determine $b^2-4ac$ and find that $b^2-4ac>0$. So we can factorise the given polynomial. Since $b^2-4ac=65$, which is not a perfect square, it is not easy to find two numbers $r$ and $s$ such that $r+s=3$ and $rs=-14$. We, therefore, use the result (2) to factorise the polynomial as

$$2x^2+3x-7=2\left[x+\frac{3+\sqrt{65}}{4}\right]\left[x+\frac{3-\sqrt{65}}{4}\right]$$

**Example 7 :** The volume of a box is given by the polynomial $V=x^3-2x^2-3x$. Express the possible dimensions of the box as linear polynomials.

**Solution :** We know that

$$\begin{aligned} V &= x^3-2x^2-3x \\ &= x(x^2-2x-3) \\ &= x\,(x-3)\,(x+1) \end{aligned}$$

Volume of a box = length × breadth × height

∴ the dimensions of the box are

$x,\ x-3,\ x+1$

The dimensions of the box are not unique. They may also be

$\frac{x}{2},\ 2x-6,\ x+1$, etc.

## Exercises 3.1

1. Factorise each of the following polynomials, if possible :

(i) $x^2-7x+12$ (ii) $6z^2-11z+3$

(iii) $2x^2+x+1$ (iv) $x^2-\sqrt{2}x+3$

(v) $x^2-(1+\sqrt{2})\,x+\sqrt{2}$ (vi) $8x^2-10xy-7y^2$

(vii) $6x^3-15x^2-36x$ (viii) $rpx^2+(2qr-p^2)\,x-2pq$, $r\neq 0, p\neq 0$

(ix) $px^2+(4p^2-3q)\,x-12pq$, $p\neq 0$

(x) $-2x^2-3x+7$ (xi) $1-4x-x^2$ (xii) $x^2+10x-2$

(xiii) $a^2b^2y^2+2abcy+c^2$, $a\neq 0, b\neq 0$

(xiv) $a^2b^2x^2-a^2x+1-b^2x$, $a\neq 0, b\neq 0$

(xv) $z^4+9z^2+8$ (xvi) $x^6+5x^3y+4y^2$

(xvii) $x^4+2x^2y^2-15y^4$ (xviii) $x^4+5x^2-6$

(xix) $x^{2n}-x^n-2$ (xx) $y^{2n}+y^n-2$

**(Hint : Put $x^n=t$)**

2. Is it possible to have a square whose area is given by $49x^2+28x+4$ and whose sides are the linear polynomials with real coefficients ? Explain.

3. Is it possible to have a rectangle whose area is given by $8x^2 - 2x - 15$ and whose sides are linear polynomials with real coefficients ? Explain.
4. The volume of a box is given by the polynomial $V = x^3 - 2x^2 - 24x$. Express the possible dimensions of the box as linear polynomials.
5. The volume of a box is given by the polynomial $V = 2x^3 + 10x^2 - 28x$. Express the possible dimensions of the box as linea polynomials.

## 3.2 Graph of a Quadratic Function

We have already learnt how to find zero (s) of a quadratic polynomial and also have learnt how to factorise a quadratic polynomial into two linear factors wherever possible. There are many problems in business, social sciences, natural sciences, life sciences etc. which can be expressed in terms of quadratic polynomials. However, the study of such problems becomes convenient and easy when we study the quadratic polynomials graphically.

Let us now consider one such problem.

A ball is thrown vertically upwards with a velocity of 25m per second. It is known that the height "$y$" in metres attained by the ball after "$x$" seconds of time is given by

$$y = -5x^2 + 25x$$

How do we draw the graph of the above function ?

Let us take some values of $x$ and calculate the corresponding values of $y$ ($x$ and $y$ cannot be negative). We are giving some of them in the following table :

| $x$ (in secconds) | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ (in metres) | 0 | 20 | 30 | 30 | 20 | 0 |

Can you plot these points and trace a smooth curve? The time-height curve is given in figure 3.1.

The graph of such a function is called a parabola. We make some observations from this graph :

At $x = 2.5$, the ball attains the maximum height, as shown by the point $C$ in figure 3.1. Thereafter, the ball starts falling downward. The point "$C$" is the turning point and is called the vertex of the parabola. One half of the parabola is a reflection (mirror image) of the other about the vertical line passing through the vertex. We say the parabola is symmetrical about this line which is called its *axis of symmetry*. Here the axis of symmetry is the vertical line whose equation is $x = 2.5$. The parabola crosses $x$-axis where the value of $y$ is zero. Thus, the values of $x$ where the parabola crosses

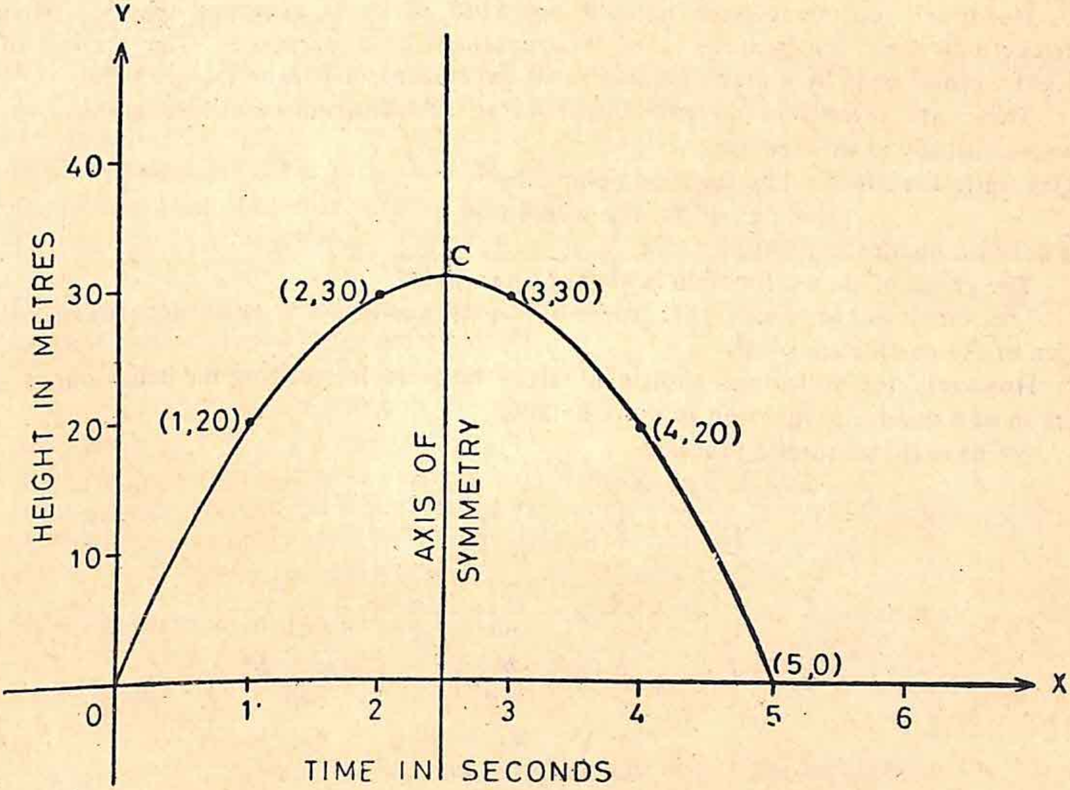



Fig. 3.1

the $x$-axis are the zeros of the polynomial $-5x^2 + 25x$. From the graph we find that 0 and 5 are the zeros of the polynomial.

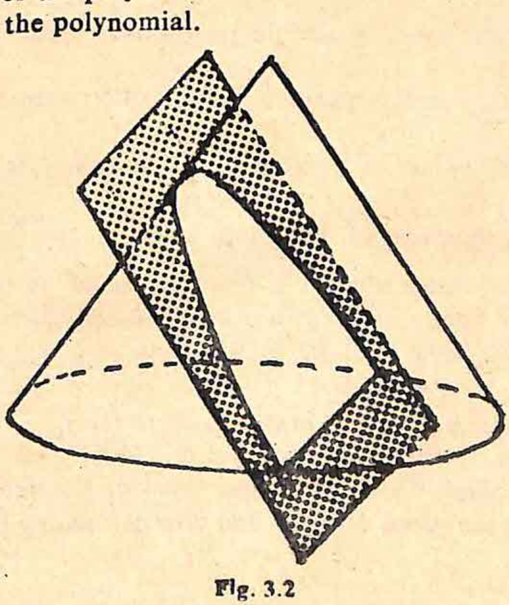


Fig. 3.2

However, you must have noticed this kind of curve at several places. When a cricket ball is hit, the path it takes is approximately a parabola. The section of a right circular cone by a plane parallel to its generator is a parabola (Fig. 3.2).

There are several important properties of such functions and their graphs, which we shall study in this section.

**Quadratic Functions :** The function defined by

$$y = ax^2 + bx + c \text{ where } a \neq 0$$

is called a quadratic function.

The graph of such a function is always a parabola.

The direction in which the graph of a quadratic function opens depends upon the sign of the co-efficient of $x^2$.

However, the following additional steps help us in studying the behaviour of the graph of a quadratic function in more details.

We have the quadratic function

$$\begin{aligned} y &= ax^2 + bx + c \qquad a \neq 0 \\ &= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] \\ &= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right] \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + a\left(\frac{4ac - b^2}{4a^2}\right) \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{-D}{4a} \qquad \text{where } D = b^2 - 4ac \end{aligned}$$

$D$ is called the discriminant of the quadratic polynomial. The quantity $\left(x + \frac{b}{2a}\right)^2$ is equal to 0 when $x = \frac{-b}{2a}$ and is positive for all other values of $x$. So this value of $x = \frac{-b}{2a}$ yeilds the least value of $y$ when $a > 0$ and the greatest value when $a < 0$. Hence $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is the point of minimum value of the function when $a > 0$ and is the point of maximum value when $a < 0$. This point is called the **Vertex** of the parabola. The parabola opens upwards if $a > 0$ and opens downwards if $a < 0$.

Thus, in order to draw a smooth graph of the quadratic function we adopt the following procedure :

**(i) Write the quadratic polynomial in the standard form.**

**(ii) Determine the zeros of the polynomial, if they exist.**

**(iii) Determine the point where the curve intersects y-axis. This can be done by putting x = 0 in the given function and then calculating the value of y.**

*(iv) **Determine the vertex, i.e.** $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$.

**(v) Prepare a table for the values of x and corresponding values of y (generally two or three points on the left and two or three points on the right of the vertex are sufficient).

**(vi) Draw a smooth curve, through these points.**

We illustrate the procedure with the help of the following examples.

**Example 1** : Find the zeros of $y = x^2 - 4$, if they exist, and graph the quadratic function.

**Solution** : I. Putting $y = 0$ and solving for $x$, the values of $x$ so obtained will tell us where the curve crosses the $x$-axis.

$$0 = x^2 - 4$$
$$0 = (x - 2)(x + 2)$$

So $x = 2$ or $x = -2$

Thus, the curve intersects the $x$-axis at $x = 2$ and at $x = -2$.

Since the co-efficient of $x^2$ is positive $(+1)$, the parabola will open upwards.

II. Putting $x = 0$ in $y = x^2 - 4$ and solving for $y$ we get

$y = -4$ i.e. the curve intersects the $y$-axis at $(0, -4)$.

III. The vertex occurs at $x = \frac{-b}{2a} = \frac{0}{2} = 0$

and at $y = \frac{4ac - b^2}{4a} = \frac{4 \times 1(-4) - 0}{4 \times 1} = -4$

i.e. the vertex is at $(0, -4)$

A table of values of $x$ and the corresponding values of $y$ of the function $y = x^2 - 4$, on either side of the vertex will help us to sketch the graph accurately. One such table is given below :

| $x$ | $-1$ | $1$ | $-3$ | $3$ | $-4$ | $4$ |
|---|---|---|---|---|---|---|
| $y$ | $-3$ | $-3$ | $5$ | $5$ | $12$ | $12$ |

The graph of the function is shown in figure 3.3.

**Example 2** : Find zeros, if any, and the vertex of the graph for the quadratic function $y = x^2 + 2x - 3$, and also draw its graph. Also graph the axis of symmetry for this function.

**Solution** : I. We have the function

$y = x^2 + 2x - 3$. The zeros of this function can be obtained by putting $y = 0$ and then solving for $x$.

---

*$y$-coordinate of the vertex can also be found by putting $x = \frac{-b}{2a}$ in the given function.

**Having traced the curve on one side of the vertex (or the axis of symmetry), we could have its mirror image in the axis to get the curve on the other side.

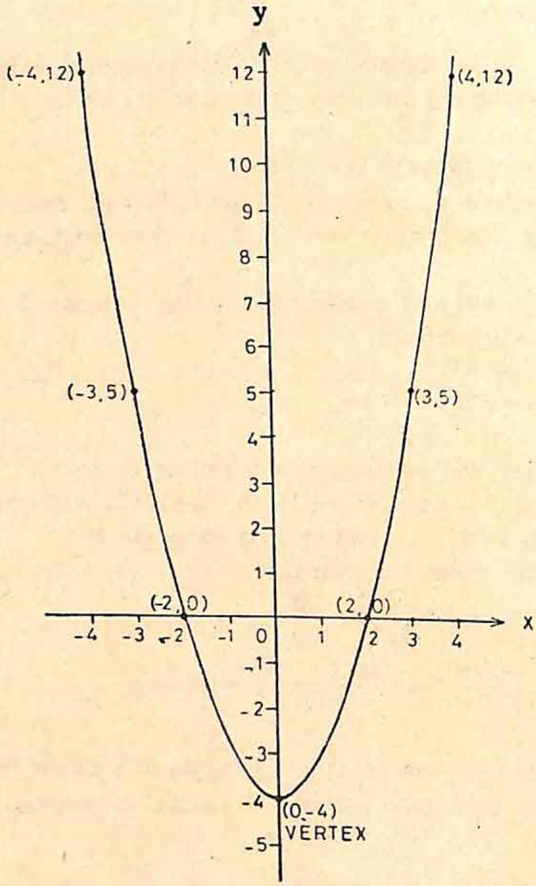



Fig. 3.3

$$0 = x^2 + 2x - 3$$
$$= (x + 3)(x - 1)$$
$$\Rightarrow x - 1 = 0 \quad \text{or } x + 3 = 0$$
i.e. $\quad x = 1 \quad$ or $x = -3$

Thus, the graph of the given function will intersect $x$-axis at (1, 0) and (−3, 0). Since the co-efficient of $x^2$ is positive (+1), the parabola will open upwards.

II. Put $x = 0$ in the given function and solve for $y$.

We have $y = -3$ i.e. the curve intersects the $y$-axis at (0, −3).

III. The $x$-coordinate of the vertex is $\frac{-b}{2a} = \frac{-2}{2 \times 1} = -1$

and the $y$-coordinate is $\frac{-D}{4a} = \frac{4 \times 1(-3) - 4}{4 \times 1} = -4$

i.e. the vertex is at (−1, −4)

Therefore, the axis of symmetry is the line $x = -1$.

IV. Plot points $(x, y)$ for some values of $x$ and the corresponding values of $y$ for the function $y = x^2 + 2x - 3$ (see table given below).

| $x$ | $-2$ | $-1$ | $0$ | $2$ | $-4$ |
|---|---|---|---|---|---|
| $y$ | $-3$ | $-4$ | $-3$ | $5$ | $5$ |

The graph of the function is shown in figure 3.4.

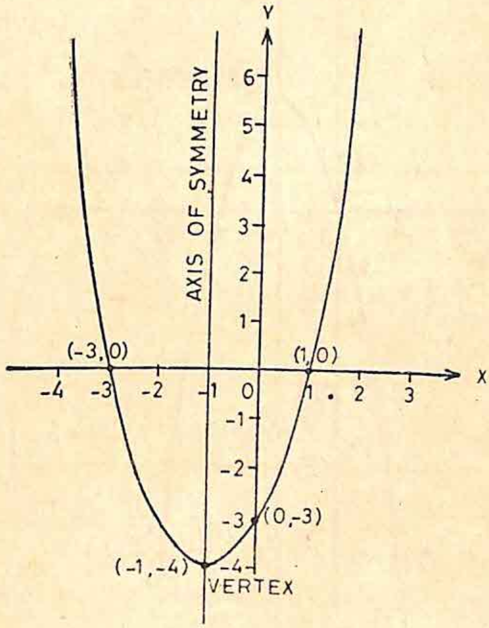



Fig. 3.4

**Example 3 :** Graph the function

$$f(x) = -2x^2 + 4x$$

Label the vertex and write its coordinates.

Also graph the axis of symmetry.

**Solution :** I. By Putting $f(x) = 0$ we determine the zeros of the given polynomial. The zeros are 0 and 2.

Since the coefficient of $x^2$ is negative $(-2)$, the parabola opens downwards.

II. Putting $x = 0$ in the given function we find that $f(0) = 0$

$\therefore$ The curve passes through the origin.

III. The coordinates of the vertex of the parabola are given by

$$x = \frac{-b}{2a} = \frac{-4}{2(-2)} = 1$$

$$y = \frac{4ac - b^2}{4a} = \frac{4(-2) \times 0 - 16}{4(-2)} = 2$$

So the vertex of the parabola is (1, 2).
Now prepare a table for the values of $x$ and the corresponding values of $y$ in order to sketch the graph accurately. The graph of the given function and its axis of symmetry are shown in figure 3.5.

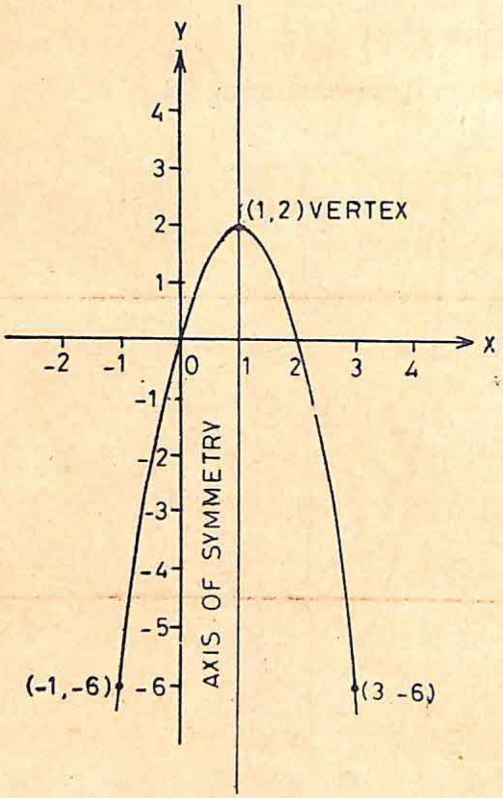



Fig. 3.5

**Example 4 :** Plot the graph of the function

$$f(x) = -2x^2 + 4x - 4$$

Also determine the point where the function is maximum.

**Solution : I.** The vertex $(x, y)$ is given by

$$x = \frac{-b}{2a} = \frac{-4}{2(-2)} = 1$$

$$y = \frac{4ac - b^2}{4a} = \frac{4(-2)(-4) - 16}{4(-2)} = -2$$

So the vertex of the parabola is at $(1, -2)$
i.e. the function is maximum at $x = 1$ and the maximum value of the given function at $x = 1$ is $-2$.

II. Discriminant of the function $-2x^2 + 4x - 4$

$$= b^2 - 4ac = 16 - 4(-2)(-4)$$
$$= 16 - 32 = -16 < 0$$

There is no zero and hence the graph does not intersect the $x$-axis.

The parabola opens downwards, because the coefficient of $x^2$ is negative.

III. $f(0) = -4$

Thus, the parabola intersects $y$-axis at $(0, -4)$.

A table for $(x, y)$ for the values of $x$ on either side of the vertex at $x = 1$ will help us to draw the graph accurately. The table is as shown below :

| $x$ | $y$ |
|---|---|
| $-1$ | $-10$ |
| 2 | $-4$ |
| 3 | $-10$ |

The graph of the function is shown in figure 3.6.

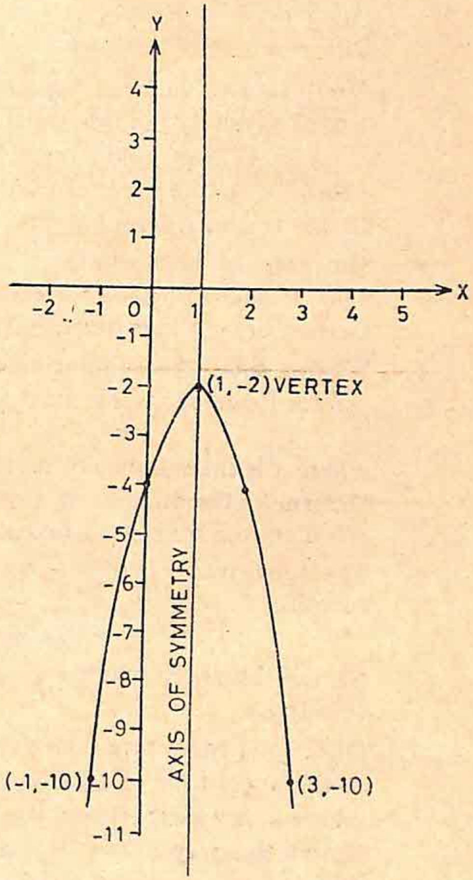



Fig. 3.6

## Exercises 3.2

1. Graph each of the following functions. Label the vertex of each parabola. Graph the axis of symmetry for each function.
   (*i*) $y = \frac{1}{2}x^2$ (*ii*) $y = 2x^2 + 4x$
   (*iii*) $y = x^2 - 2x + 3$ (*iv*) $y = -x^2 + 4x - 3$
2. Graph each of the following quadratic functions. Determine the points of maximum or the points of minimum values and axis of symmetry in each case.
   (*i*) $y = x^2 + 2x + 5$ (*ii*) $y = -2x^2 - 8x$
   (*iii*) $y = -2x^2$ (*iv*) $y = x^2 - 6x + 2$
3. A ball thrown vertically upward attains the height "$y$" (in metres) above the ground given by the function
   $y = 112x - 16x^2$
   where "$x$" is the time taken in seconds. Determine the time taken when the ball attains the maximum height.
4. The rate of photosynthesis '$y$' for a certain plant depends on the intensity of light '$x$' according to the relation $y = 90\ (3x - x^2)$
   Determine, without drawing the graph, the intensity that gives the maximum rate.
5. When a small manufacturing company produces and sells from 10 to 60 units of certain items per week, its revenue in Rs is given by
   $$y = -x^2 + 100x$$
   where $x$ is the number of units produced.
   Determine the number of units for which the revenue is maximum and hence calculate the maximum revenue.
6. The sensitivity "$y$" to a particular drug is related to the dosage "$x$" by the function
   $$y = 10x - x^2$$
   Sketch the graph of this function and determine what dosage gives maximum sensitivity.
7. One of the early results in psychology relating the magnitude of a stimulus "$x$" to the magnitude of a response "$y$" is expressed by the equation $y = kx^2$ where $k$ is an experimental constant.
   Sketch the graph for $k = 1$ and $k = -2$.
8. The electric power $y$ (in watts) in a 240 volt line having a resistance of 20 ohms is given by the formula
   $y = 240x - 20x^2$
   where $x$(in amps) is the current flowing in the line.
   Determine how much current must flow to attain the maximum power. What is the maximum power that can be delivered in this circuit ?

9. A manufacturer finds that the cost per chair "$y$" of manufacturing a certain type of chair is
$y = x^2 - 20x + 200$ when $x$ chairs ($x$ can represent from 5 to 200 chairs) are produced in a given day.
(a) Graph the given function
(b) Locate on the graph, the cost per chair when 10 chairs are produced per day.

## 3.3 Quadratic Equations

In section 3.2 we have seen that the height of a ball when thrown vertically upward is given by $y = -5x^2 + 25x$, where $y$ is the height in metres and $x$ is the time in seconds after the ball is thrown upwards. Now to find the time (number of seconds), before the ball returns to earth, we may write $0 = -5x^2 + 25x$ and solve it for the value or values of $x$. Now we may check whether this equation is true for $x = 0$ or not? Do you think that this represents the time taken by the ball to return to earth? If not, what does it represent? Of course this represents the time when the ball is thrown vertically upwards. Is this equation true for $x = 5$? Then $x = 5$ is the time in seconds, when the ball reaches the earth.

Examine the equation $0 = -5x^2 + 25x$. The R.H.S. of this equation is a polynomial of degree two. Thus, we call the equation $0 = -5x^2 + 25x$ a quadratic equation. Each of the following equations is also a quadratic equation.

$$3 - 4x^2 = 0,\ y^2 = 4,\ 2x^2 - 18x + 3 = 0,\ 6x^2 = 4x - 3$$

The standard form of the quadratic equation in one variable is $ax^2 + bx + c = 0$, where $a \neq 0$.

**3.31 Roots of the Quadratic Equation** : We have factorised the polynomial $ax^2+bx + c$ in the section 3.1 as follows :

*Case I* : $b^2 - 4ac > 0$

$$ax^2 + bx + c = a\left[x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right]\left[x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right]$$

We see that $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

are the zeros of the polynomial $ax^2 + bx + c$.

Hence $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ are the two roots of the equation $ax^2 + bx + c = 0$.

Thus, the solution set is

$$\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a},\ \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}$$

*Case II* : $b^2 - 4ac = 0$

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

This shows that $x = -\frac{b}{2a}$ is the only zero of the polynomial or we say that the quadratic equation $ax^2 + bx + c = 0$ has a pair of coincident roots.

**The solution set in** this case is$\{-\frac{b}{2a}\}$.

*Case III* : $b^2 - 4ac < 0$. We have seen in section 3.1 that $ax^2 + bx + c$ has no linear factors. Hence by factor theorem the equation $ax^2 + bx + c = 0$ has no real roots or we say that the solution set is $\phi$.

**Alternate Method** : However, we can solve the quadratic equation independent of the method discussed above. The method which we now follow is called method of completing squares.

We have $\quad ax^2 + bx + c = 0,\ a \neq 0$

Multiplying both sides of this equation by $4a$ we get

$$4a^2x^2 + 4abx + 4ac = 0$$

$$\Leftrightarrow \quad 4a^2x^2 + 4abx = -4ac$$

Adding $b^2$ to both sides of the equation we get

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

$$\Leftrightarrow \quad (2ax + b)^2 = b^2 - 4ac$$

*Case I* : *If* $b^2 - 4ac \geqslant 0$

$$\text{then } 2ax + b = \pm \sqrt{b^2 - 4ac}$$

$$2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence the roots of the quadratic equation are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and the solution set is $\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a},\ \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}$

*Case II* : *If* $b^2 - 4ac < 0$

then $(2ax + b)^2 = b^2 - 4ac$ is not true for any real value of $x$, as the left hand side cannot be negative. Hence the equation has no roots.

**Example 1** : Find the roots of the equation

$$x^2 - 5x + 6 = 0$$

**Solution** : Here

$$a = 1,\ b = -5 \text{ and } c = 6$$

$$D = b^2 - 4ac = (-5)^2 - 4 \times 1 \times 6$$
$$= 25 - 24 = 1$$

$\Rightarrow \quad D > 0$

$\therefore$ equation has two distinct roots

Now $\quad x^2 - 5x + 6 = 0$

$$\Leftrightarrow \quad x^2 - 3x - 2x + 6 = 0$$

$\Leftrightarrow \quad x(x-3)-2(x-3)=0$

$\Leftrightarrow \quad (x-3)(x-2)=0$

$\Leftrightarrow \quad x-3=0 \text{ or } x-2=0$

$\Leftrightarrow \quad x=3 \text{ or } x=2$

Hence the roots of the equation are 3 and 2.

**Example 2**: Find the solution set of the equation

$$2y^2-6y+3=0$$

**Solution**: Here $a=2$, $b=-6$ and $c=3$

$\therefore \quad D=12>0$

$\therefore$ the equation has two distinct roots.

Let us apply the quadratic formula

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \text{ to find the solution set.}$$

Then $$y=\frac{-(-6)\pm\sqrt{12}}{2\times 2}$$

$$=\frac{6\pm\sqrt{12}}{4}=\frac{3\pm\sqrt{3}}{2}$$

Hence the solution set is $\left\{\frac{3+\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}\right\}$

**Example 3**: Find the roots of the equation

$$4t^2+4t+1=0$$

**Solution**: Here $a=4$, $b=4$ and $c=1$

$\therefore \quad D=b^2-4ac=16-4\times 4\times 1=0$

i.e. $\quad D=0$

$\therefore \quad t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=-\frac{b}{2a}=-\frac{4}{8}=-\frac{1}{2}$

Hence the root is $-\frac{1}{2}$.

**Example 4**: Find the solution set of the equation

$$z^2-3z+11=0$$

**Solution**: Here $a=1$, $b=-3$ and $c=11$

$$D=b^2-4ac=(-3)^2-4\times 1\times 11$$
$$=+9-44=-35<0$$

Since $D<0$, the solution set of the given equation is $\phi$.

**Example 5**: Find the roots of the equation

$$\sqrt{7}x^2-6x-13\sqrt{7}=0$$

**Solution**: Here $a=\sqrt{7}$, $b=-6$ and $c=-13\sqrt{7}$

$$D=b^2-4ac=36+364=400>0$$

Now we apply the formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{400}}{2\sqrt{7}}$$
$$= \frac{3 \pm 10}{\sqrt{7}}$$

Hence the roots are

$$\frac{13\sqrt{7}}{7} \text{ and } -\sqrt{7}$$

**Example 6 :** Find the solution set of the equation
$$10x^2 + 3bx + a^2 - 7ax - b^2 = 0$$

**Solution :** Let us re-arrange the given equation in the standard form. We have
$$10x^2 + (3b - 7a)x + (a^2 - b^2) = 0$$
Here $A = 10$, $B = 3b - 7a$ and $C = a^2 - b^2$.

so $$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$
$$= \frac{-(3b - 7a) \pm \sqrt{(3b - 7a)^2 - 4 \times 10\,(a^2 - b^2)}}{20}$$
$$= \frac{-(3b - 7a) \pm \sqrt{9a^2 - 42ab + 49b^2}}{20}$$
$$= \frac{-(3b - 7a) \pm \sqrt{(3a - 7b)^2}}{20}$$
$$= \frac{-(3b - 7a) \pm (3a - 7b)}{20}$$

So the roots are $\frac{a - b}{2}$ and $\frac{a + b}{5}$

Hence the solution set is $\left\{\frac{a - b}{2}, \frac{a + b}{5}\right\}$

**Example 7 :** For what values of "$k$" the equation
$x^2 + 4x + k = 0$ has real roots ?

**Solution :** Comparing the given equation with the standard form of the equation we have
$$a = 1,\ b = 4 \text{ and } c = k$$
$$\therefore\ D = b^2 - 4ac = 16 - 4k$$
We know that the quadratic equation has real roots only
if $D = b^2 - 4ac \geqslant 0$
$$\therefore\ 16 - 4k \geqslant 0$$
or $$k \leqslant 4$$
The given equation has real roots only when $k \leqslant 4$.

## Exercises 3.3

1. Solve the following equations, if possible :
   (i) $x^2 - 8x + 15 = 0$ (ii) $3x^2 + 2x - 5 = 0$
   (iii) $5z^2 + 12z + 10 = 0$ (iv) $5x^2 = x + 1$
   (v) $x^2 - 4x + 7 = 0$
2. Find the roots, if possible, of the following equations :
   (i) $4y^2 = 1$ (ii) $\frac{4}{3}x^2 - 2x + \frac{3}{4} = 0$
   (iii) $25x(x + 1) = -4$ (iv) $x^2 + (a - b)x = ab$
   (v) $(3x + a)(3x + b) = ab$
3. Find the solution set of each of the following equations :
   (i) $t^2 - 4t + 10 = 0$ (ii) $\sqrt{3}x^2 + 11x + 6\sqrt{3} = 0$
   (iii) $4\sqrt{3}x^2 + 5x - 2\sqrt{3} = 0$ (iv) $px^2 + (4p^2 - 3q)x - 12pq = 0,\ p \neq 0$
   (v) $(m + n)^2 x^2 + (m + n)x - 2 = 0\ ,\ m + n \neq 0$
4. Determine $k$ so that the following equations have coincident roots :
   (i) $x^2 + kx + 4 = 0$ (ii) $2x^2 - kx + 1 = 0$
   (iii) $kz^2 - z = 2$ (iv) $y^2 + 2y + k + 1 = 0$
   (v) $t^2 + k^2 = 2(k + 1)t$
5. Determine $k$ so that the equation
   $x^2 - 4x + k = 0$ has
   (i) two roots (ii) coincident roots
6. Determine $\lambda$ so that the equation
   $x^2 + \lambda x + \lambda + 1.25 = 0$ has
   (i) two distinct roots (ii) two coincident roots

### 3.4 Equations Reducible to Quadratic Form

At times, we come across equations which by proper substitution or after simplification, can be transformed into quadratic equations. Such equations are called equations reducible to quadratic form. Let us consider some examples to illustrate the solution of such equations :

**Example 1** : Solve the equation

$$z^4 + 3z^2 - 4 = 0 \text{ if possible.}$$

**Solution** : We find that the given equation is a fourth degree equation in $z$.

We can re-write it as

$$(z^2)^2 + 3z^2 - 4 = 0$$

Put $z^2 = x$

Now we have

$$x^2 + 3x - 4 = 0$$

or $(x + 4)(x - 1) = 0$

$\Rightarrow$ $x = -4$ or $x = 1$

Replacing $x$ by $z^2$ we have

$$z^2 = -4 \quad \text{or} \quad z^2 = 1$$

Since there is no real root of the equation $z^2 = -4$, so we discard this equation and solve the equation $z^2 = 1$ for its real roots.

So we get $z = \pm 1$

$\therefore$ required solution set is $\{-1, 1\}$

The student may verify that these values satisfy the given equation.

**Example 2 :** Solve for $x$ :

$$x^3 - 7x + 6 = 0$$

**Solution :** By factor theorem $x - 1$ is a factor of the expression $x^3 - 7x + 6$

$\therefore \quad x^3 - 7x + 6 = (x-1)(x^2 + x - 6) = 0$

$\qquad\qquad = (x-1)(x+3)(x-2) = 0$

$x = 1, x = -3$ or $x = 2$

$\therefore$ the solution set is $\{1, 2, -3\}$

**Example 3 :** Solve

$$\left(x^2 + \frac{1}{x^2}\right) + 4\left(x + \frac{1}{x}\right) + 6 = 0; \qquad x \neq 0$$

**Solution :** We have

$$\left(x^2 + \frac{1}{x^2}\right) + 4\left(x + \frac{1}{x}\right) + 6 = 0 \qquad \ldots(i)$$

Put $\quad x + \dfrac{1}{x} = y \qquad \ldots(ii)$

then $\quad x^2 + \dfrac{1}{x^2} = y^2 - 2 \qquad \ldots(iii)$

Substituting $(ii)$ and $(iii)$ in $(i)$ then equation $(i)$ reduces to

$y^2 - 2 + 4y + 6 = 0$

*i.e.* $\quad y^2 + 4y + 4 = 0$

or $\quad (y+2)^2 = 0$

*i.e.* $\quad y = -2 \qquad \ldots(iv)$

Now replacing $y$ by $x + \dfrac{1}{x}$ in $(iv)$ we have

$$x + \frac{1}{x} = -2$$

or $\quad x^2 + 2x + 1 = 0$

or $\quad (x+1)^2 = 0$

*i.e.* $\quad x = -1$

Thus, $-1$ is the only solution of the given equation.

**Example 4 :** Solve for $x$

$$3^{x-1} + 3^{1-x} = 2$$

**Solution :** We can write the given equation as

$$3^{x-1} + \frac{1}{3^{x-1}} = 2 \qquad \ldots(i)$$

Now put $3^{x-1} = y$. Then equation (*i*) reduces to $y + \frac{1}{y} = 2$

or $y^2 - 2y + 1 = 0$

*i.e.* $(y - 1)^2 = 0$

*i.e.* $y = 1$ ...(*ii*)

Replacing $y$ by $3^{x-1}$ in (*ii*) we get

$$3^{x-1} = 1$$

*i.e.* $3^{x-1} = 3^0$

Since the bases are same on both sides of the equation so equating their exponents we get

$$x - 1 = 0$$

$\therefore$ The solution of the given equation is 1.

The reader may check that the equation is satisfied by $x = 1$.

**Example 5 :** Solve the equation

$$x + \sqrt{x - 2} = 8$$

**Solution :** We are given the equation as

$$x + \sqrt{x - 2} = 8$$

We first isolate the radical by writing the equation in the equivalent form

$$\sqrt{x - 2} = 8 - x$$

Now squaring both sides we get

$$x - 2 = (8 - x)^2$$
$$= 64 - 16x + x^2$$

On simplifying we get

$$x^2 - 17x + 66 = 0$$
$$(x - 11)(x - 6) = 0$$
$$x - 11 = 0 \text{ or } x - 6 = 0$$
$$x = 11 \text{ or } x = 6$$

CHECK : When $x = 11$ we have

$$x + \sqrt{x - 2} = 11 + \sqrt{11 - 2}$$

Since LHS $\neq$ RHS,

$x = 11$ is not a root

When $x = 6$, we have

$$\text{LHS} = x + \sqrt{x - 2} = 6 + \sqrt{6 - 2}$$
$$= 6 + 2$$
$$= 8$$
$$= \text{RHS}$$

Hence the solution set is $\{6\}$.

*Note* : As the process of squaring is not reversible, sometimes we get values which do not satisfy the given equation and as such are not the roots. Therefore in all situations like this, a check is absolutely essential before the final solution is stated.

**Example 6 :** Solve for $x$, the equation

$$\sqrt{3x-5}+\sqrt{x+2}=3$$

**Solution :** The given equation is

$$\sqrt{3x-5}+\sqrt{x+2}=3 \qquad \ldots (i)$$

This can be rewritten as

$$\sqrt{3x-5}=3-\sqrt{x+2} \qquad \ldots (ii)$$

Squaring both sides of equation (*ii*) we get

$$3x-5=9+(x+2)-6\sqrt{x+2}$$

Simplifying we get

$$2x-16=-6\sqrt{x+2}$$

*i.e.* $$x-8=-3\sqrt{x+2} \qquad \ldots (iii)$$

Squaring (*iii*) we get

$$(x-8)^2=9(x+2)$$

*i.e.* $$x^2-25x+46=0$$

*i.e.* $$(x-23)(x-2)=0$$

*i.e.* $$x=23 \text{ or } x=2$$

By putting these values of $x$ in (*i*) and verifying we find that only $x=2$ satisfies the given equation.

Thus, the solution set is $\{2\}$.

## Exercises 3.4

Solve the following equations :

1. $x^4-13x^2+36=0$
2. $9z^4+25=30z^2$
3. $x^3-4x^2+x+2=0$
4. $x^3-6x^2+11x-6=0$
5. $x^4-2x^3-2x^2+2x+1=0$
6. $\left(x^2+\frac{1}{x^2}\right)-3\left(x-\frac{1}{x}\right)-2=0,\ x\neq 0$
7. $2\left(x^2+\frac{1}{x^2}\right)+\left(x-\frac{1}{x}\right)-10=0,\ x\neq 0$
8. $2\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-1=0,\ x\neq 0$
9. $2^{2x}+2^{x+1}=4-2^x$
10. $16\cdot 4^{x+2}-16\cdot 2^{x+1}+1=0$
11. $2^{2x+3}=65(2^x-1)+57$

12. $\left(\frac{2x-3}{x-1}\right) - 4\left(\frac{x-1}{2x-3}\right) = 3\,;\ x \neq 1,\ x \neq \frac{3}{2}$

13. $8\sqrt{\frac{x}{x+3}} - \sqrt{\frac{x+3}{x}} = 2\,;\ x \neq -3,\ x \neq 0$

14. $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}\,;\ x \neq 0,\ x \neq 1$

15. $\sqrt{4x^2 + x - 2} + 1 = 2x$

16. $x - \sqrt{25 - x^2} = 1$

17. $2\sqrt{x-1} - \sqrt{5+2x} = 1$

18. $\sqrt{2x+5} + \sqrt{x-2} = -2$

19. $\sqrt{(x-1)(x-2)} + \sqrt{(x-3)(x-4)} = 2$

20. $\sqrt{2x^2 + x + 3} + \sqrt{2x^2 + x - 6} = 3$

## 3.5 Solutions of Problems involving Quadratic Equations

There are many word problems which can be solved by means of quadratic equations. The method for setting up the necessary equation is more or less the same as for the word problems which we studied in the previous classes. Sometimes only one root of the quadratic equation has a meaning for the problem, Any root not satisfying the conditions of the given problem must be rejected. We consider such word problems which involve applications of quadratic equations.

**Example 1 :** The sum of the two numbers is 12. The sum of their squares is 80. Determine each number.

**Solution :** Let one number be $x$ and the other number be $y$.

The sum of the numbers is 12.

$$x + y = 12 \quad \text{...}(i)$$

The sum of their squares is

$$x^2 + y^2 = 80 \quad \text{...}(ii)$$

Substituting the value of $y$ from $(i)$ in $(ii)$ we get

$$x^2 + (12 - x)^2 = 80$$
$$\Leftrightarrow \quad x^2 - 12x + 32 = 0$$
$$\Leftrightarrow \quad (x-4)(x-8) = 0$$
$$\Leftrightarrow \quad x - 4 = 0 \text{ or } x - 8 = 0$$
$$\Leftrightarrow \quad x = 4 \text{ or } x = 8$$

Now from $(i)$ when $x = 4$ we get $y = 8$

or when $x = 8$ we get $y = 4$

The numbers are 4 and 8.

CHECK : According to the given conditions

sum of the numbers should be 12.

Now $\quad 4 + 8 = 12$

$\therefore$ The condition is satisfied.

Also sum of their squares should be 80.

Now $4^2 + 8^2 = 80$

Therefore this condition is also satisfied.

Hence the solution is correct.

**Example 2 :** In the game of cards, Subhash scored 3 points more than twice the number of points Promila scored. If the product of their scores was 65 points, how many points did each score ?

**Solution :** Let the number of points scored by Promila be $x$. Then the number of points scored by Subhash is $2x + 3$.

Since the product of their scores is 65 we have

$$x(2x + 3) = 65$$

$$\Leftrightarrow \quad 2x^2 + 3x = 65$$

$$\Leftrightarrow \quad 2x^2 + 3x - 65 = 0$$

$$\Leftrightarrow \quad (2x + 13)(x - 5) = 0$$

$$\Leftrightarrow \quad 2x + 13 = 0 \text{ or } x - 5 = 0$$

$$\Leftrightarrow \quad x = \frac{-13}{2} \text{ or } x = 5$$

Negative scores in the game are not permissible.

$\therefore$ Promila scored 5 points and Subhash scored 13 points in the game.

**Example 3 :** A model rocket is shot straight up. Its height $y$ in metres from the ground level after $x$ seconds is given by the quadratic function

$$y = -5x^2 + 200x$$

Determine in how many seconds will the model rocket be 1875 metres above the ground.

**Solution :** Here we have $y = 1875$ metres

Substituting in the given quadratic function

we have $\quad 1875 = -5x^2 + 200x$

i.e. $\quad x^2 - 40x + 375 = 0 \qquad \ldots(i)$

i.e. $\quad (x - 15)(x - 25) = 0$

i.e. $\quad x - 15 = 0$ or $x - 25 = 0$

$\Rightarrow \quad x = 15$ or $x = 25$

Here both the values of $x$ are possible. Why ?

Can you plot the graph of the function $(i)$ ? Do it and read the values of $y$ at $x = 15$ and at $x = 25$.

Are these the same ? Why ?

**Example 4 :** The hypotenuse of a right angled triangle is one metre less than twice the shortest side. If the base of the triangle is one metre more than the shortest side, determine the lengths of the three sides of the triangle.

**Solution :** Let the length of the shortest side be $x$ metres. Then the length of the base will be $(x + 1)$ metres and the hypotenuse will be $(2x - 1)$ metres.

Now according to the Pythagoras theorem,

$$(2x - 1)^2 = x^2 + (x + 1)^2$$

i.e. $4x^2 - 4x + 1 = x^2 + x^2 + 2x + 1$

Simplifying we get

$$2x^2 - 6x = 0$$
$$x^2 - 3x = 0$$
$$x(x - 3) = 0$$

i.e. $x = 0$ or $x = 3$

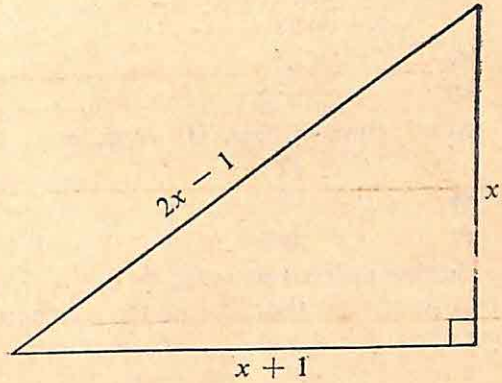


**Fig. 3.7**

Since the length of the side cannot be zero we reject the value of $x = 0$.

$\therefore$ the length of the three sides of a right angled triangle will be 3 metres, 4 metres and 5 metres.

**Example 5 :** Swati can row her boat 5 km per hour in still water. If it takes her one hour longer to row the boat 5.25 km upstream than to return downstream, find the speed of the stream.

**Solution :** Let the speed of the stream be $x$ km/h. Then the speed of the boat going upstream is $(5 - x)$ km/h and the speed of the boat going downstream is $(5 + x)$ km/h. According to the conditions of the problem we have

$$\frac{5.25}{5 - x} - 1 = \frac{5.25}{5 + x}$$

Simplifying we get

$$\frac{21}{2}x = 25 - x^2$$

i.e. $2x^2 + 21x - 50 = 0$

i.e. $(2x + 25)(x - 2) = 0$

$\Leftrightarrow$ either $2x + 25 = 0$ or $x - 2 = 0$

i.e. $x = -\frac{25}{2}$ or $x = 2$

Since the speed of the stream cannot be negative

$\therefore$ $x = -\frac{25}{2}$ is rejected.

Thus, the speed of the stream is 2 km per hour.

**Example 6 :** The sum of the squares of two numbers is 41. The difference of the squares of these numbers is 9. What are the numbers ?

**Solution :** Let the two numbers be $x$ and $y$. Then according to the question, we have

$$x^2 + y^2 = 41 \quad \text{...(i)}$$

and

$$x^2 - y^2 = 9 \quad \text{...(ii)}$$

Adding (*i*) and (*ii*) we get

$$2x^2 = 50$$
$$\Leftrightarrow \quad x^2 = 25$$
$$\Leftrightarrow \quad x = \pm 5$$

Subtracting (*ii*) from (*i*) we have

$$2y^2 = 32$$
$$\Leftrightarrow \quad y^2 = 16$$
$$\Leftrightarrow \quad y = \pm 4$$

So the ordered pairs (5, 4), (− 5, 4), (5, − 4) and (− 5, − 4), all satisfy the given conditions. So these are all the members of the solution set.

## Exercises 3.5

1. Find two consecutive odd integers, the sum of whose squares is 202.
2. The sum of two numbers is 40. Find the numbers if the sum of their reciprocals is $\frac{2}{5}$.
3. There are three consecutive integers such that the square of the first increased by the product of the other two gives 154. What are the integers ?
4. A two-digit number is such that the product of the digits is 8. When 18 is added to the number, the digits interchange their places. Determine the number.
5. In a school auditorium, the number of seats in each row is 8 fewer than the number of rows. How many seats are in each row if there are in all 609 seats in the auditorium ?
6. The number of straight lines $y$ that can connect $x$ points is given by the equation
$$y = \frac{x}{2}(x - 1)$$
How many points does a figure have if only 15 lines can be drawn connecting them ?
7. The hypotenuse of a right angled triangle is 25 metres long. The difference between the lengths of the other two sides is 5 metres. Find the lengths of the other sides of the triangle.
8. Vikram wishes to fit three rods together in the shape of a right angled triangle. The hypotenuse is to be 2 cm longer than the base and 4 cm longer than the altitude. How long should he cut each rod ?
9. Two trains leave New Delhi Railway Station. The first train travels due-west and the second train due-north. The first train travels 5 km/hr faster than the second train. If after two hours they are 50 km apart, find the average speed of each train.

10. Varun wishes to start a 100 sq. m rectangular vegetable garden. Since he has only 30 metres barbed wire for fencing, he fences three sides of the rectangle, letting his garage wall act as the fourth side. How wide is his vegetable garden ?
11. Ashwani can row downstream 3 km/hr faster than he can row upstream. He finds that he can row one km upstream and back to his starting point in one hour. What is his speed while going downstream ?
12. A motor-boat moving at 9 km/hr in still water goes 12 km downstream and comes back in total of 3 hours. Determine the speed of the water.
13. Ashu is $m$ years old while his mother Mrs Veena is $m^2$ years old. 5 years hence Mrs Veena will be three times as old as Ashu. Find their present ages.
14. The sum of the ages of a father and son is 45 years. Five years ago the product of their ages was four times the father's age at that time. Find their present ages.
15. The difference of the squares of two numbers is 45. The square of the smaller number is 4 times the larger number. What are the numbers ?
16. The hypotenuse of a right angled triangle is $3\sqrt{10}$ cm. If the smaller leg is tripled and the longer leg is doubled, new hypotenuse will be $9\sqrt{5}$ cm. How long is each side of the triangle ?
17. If twice the area of a smaller square is subtracted from the area of the larger square, the difference will be 14 sq. cm. However, if twice the area of the larger square is added to three times the area of the smaller square, the sum will be 203 sq. cm. How long is the side of each square ?

CHAPTER 4

# Factorisation, Highest Common Factor and Least Common Multiple

## 4.1 Introduction

A polynomial in this chapter will always mean a polynomial with real coefficients. We have learnt in chapter 3 how to factorise quadratic polynomials into two linear factors. We may emphasise again that sometimes it is not possible to factorise a quadratic polynomial into two linear factors. This happens precisely when its discriminant is negative. We came across examples of some such polynomials in chapter 3.

In this chapter, we shall learn how to factorise some polynomials of third and fourth degrees. At the outset, it may be stated, without proof, that a cubic polynomial can always be factorised so that at least one of the factors is linear and a polynomial of degree four can always be factorised into two quadratic factors, which may or may not be factorisable into linear factors. We always try to factorise any polynomial into linear factors, if possible.

We now obtain the factors of some special polynomials, along with some examples to illustrate :

## 4.2 Factorisation of $x^3 + y^3$

Let $\quad f(x) = x^3 + y^3$

We find that $f(-y) = (-y)^3 + y^3 = 0$

Thus $x - (-y) = x + y$ is a factor of the polynomial $x^3 + y^3$.

We now divide $x^3 + y^3$ by $x + y$ to get the other factor.

$$
\begin{array}{r|l}
 & x^2 - xy + y^2 \\
\hline
x + y & x^3 + y^3 \\
 & \underline{x^3 + x^2y} \\
 & -x^2y + y^3 \\
 & \underline{-x^2y - xy^2} \\
 & xy^2 + y^3 \\
 & \underline{xy^2 + y^3} \\
 & 0
\end{array}
$$

Thus, $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

We now examine $x^2 - xy + y^2$ for further factorisation.

Considering the factor $x^2 - xy + y^2$ as a quadratic in $x$, we have

$$\text{Discriminant } D = (-y)^2 - 4\cdot 1\cdot y^2$$
$$= -3y^2$$

which is negative.

Thus $x^2 - xy + y^2$ has no linear factors and so

**Formula 1:** $\mathbf{x^3 + y^3 = (x + y)(x^2 - xy + y^2)}$

We now illustrate the use through some examples.

**Example 1:** Find the factors of $x^3 + 27y^3$

**Solution:** $x^3 + 27y^3$

$$= x^3 + (3y)^3$$
$$= (x + 3y)(x^2 - 3xy + 9y^2) \text{ (using formula 1)}$$

**Example 2:** Factorise $64x^3 + 125y^3$

**Solution:** $64x^3 + 125y^3$

$$= (4x)^3 + (5y)^3$$
$$= (4x + 5y)(16x^2 - 20xy + 25y^2)$$

**Example 3:** Find the factors of $3\sqrt{3}y^3 + 8$

**Solution:** $3\sqrt{3}\,y^3 + 8$

$$= (\sqrt{3}\,y)^3 + 2^3$$
$$= (\sqrt{3}\,y + 2)(3y^2 - 2\sqrt{3}\,y + 4)$$

**Example 4:** Factorise $8x^3 + \dfrac{y^3}{125}$

**Solution:** $8x^3 + \dfrac{y^3}{125}$

$$= (2x)^3 + \left(\frac{y}{5}\right)^3$$
$$= \left(2x + \frac{y}{5}\right)\left(4x^2 - \frac{2xy}{5} + \frac{y^2}{25}\right)$$

**Example 5:** Find the factors of $a^{\frac{3}{2}} + \dfrac{b^{\frac{3}{2}}}{8}$

**Solution:** $a^{\frac{3}{2}} + \dfrac{b^{\frac{3}{2}}}{8}$

$$= \left(a^{\frac{1}{2}}\right)^3 + \left(\frac{b^{\frac{1}{2}}}{2}\right)^3$$
$$= \left(a^{\frac{1}{2}} + \frac{b^{\frac{1}{2}}}{2}\right)\left(a - \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{2} + \frac{b}{4}\right)$$

## Exercises 4.1

Factorise the following :

1. $x^3 + 1$
2. $216 + a^3$
3. $a^4 + a$
4. $8x^3 + 125y^3$
5. $27a^3 + 7\sqrt{7}\,b^3$
6. $\frac{a^3}{8} + 5\sqrt{5}\,b^3$
7. $a^3 + \frac{y^3}{343}$
8. $\frac{a^3}{2\sqrt{2}} + \frac{b^3}{3\sqrt{3}}$
9. $8(x + y)^3 + 27(x - y)^3$
10. $y^6 + z^6$
11. $125y^{\frac{3}{4}} + z^{\frac{3}{4}}$
12. $5\sqrt{5}\,x^{\frac{3}{2}} + 8y^{\frac{3}{2}}$
13. $(a + 2b)^3 + (2a + b)^3$
14. $xy^3 + 729x^4$

### 4.3 Factorisation of $x^3 - y^3$

Let us denote $x^3 - y^3$ by $f(x)$.

Then $f(y) = y^3 - y^3 = 0$ and hence, by Factor Theorem, $x - y$ is a factor of $x^3 - y^3$.

Dividing $x^3 - y^3$ by $x - y$, we get the other factor to be $x^2 + xy + y^2$.

Exactly as in section 4·2, we can show that $x^2 + xy + y^2$ cannot be factored into linear factors. Thus, we have :

**Formula 2 :** $\mathbf{x^3 - y^3 = (x - y)(x^2 + xy + y^2)}$

Aliter : Here is another method of finding factors of $x^3 - y^3$.

$$
\begin{aligned}
x^3 - y^3 &= x^3 + (-y)^3 \\
&= \{x + (-y)\}\{x^2 - x(-y) + (-y)^2\} \text{ (by formula 1)} \\
&= (x - y)(x^2 + xy + y^2)
\end{aligned}
$$

The use of this formula is also now illustrated through the examples which follow.

**Example 1 :** Factorise $64a^3 - b^3$

**Solution :** $64a^3 - b^3$

$$
\begin{aligned}
&= (4a)^3 - b^3 \\
&= (4a - b)(16a^2 + 4ab + b^2) \text{ (by using formula 2)}
\end{aligned}
$$

**Example 2 :** Find the factors of $125x^3 - 27y^3$

**Solution :** $125x^3 - 27y^3$

$$
\begin{aligned}
&= (5x)^3 - (3y)^3 \\
&= (5x - 3y)(25x^2 + 15xy + 9y^2)
\end{aligned}
$$

**Example 3 :** Factorise $16\sqrt{2}\,a^3 - \frac{y^3}{27}$

**Solution :** $16\sqrt{2}\,a^3 - \frac{y^3}{27}$

$$= (2\sqrt{2}\,a)^3 - \left(\frac{y}{3}\right)^3$$

$$= \left(2\sqrt{2}\,a - \frac{y}{3}\right)\left(8a^2 + \frac{2ay\sqrt{2}}{3} + \frac{y^2}{9}\right)$$

**Example 4 :** Factorise $3\sqrt{3}\,y^3 - 8z^6$

**Solution :** $3\sqrt{3}\,y^3 - 8z^6$

$$= (\sqrt{3}\,y)^3 - (2z^2)^3$$

$$= (\sqrt{3}\,y - 2z^2)(3y^2 + 2\sqrt{3}\,yz^2 + 4z^4)$$

**Example 5 :** Find the factors of $8y^{\frac{3}{5}} - \frac{z^{\frac{3}{5}}}{27}$

**Solution :** $8y^{5} - \frac{z^{\frac{3}{5}}}{27}$

$$= \left(2y^{\frac{1}{5}}\right)^3 - \left(\frac{z^{\frac{1}{5}}}{3}\right)^3$$

$$= \left(2y^{\frac{1}{5}} - \frac{z^{\frac{1}{5}}}{3}\right)\left(4y^{\frac{2}{5}} + \frac{2}{3}\,y^{\frac{1}{5}}z^{\frac{1}{5}} + \frac{z^{\frac{2}{5}}}{9}\right)$$

## Exercises 4.2

Find the factors in each of the following cases :

1. $a^3 - 8b^3$
2. $2a^3 - 16$
3. $27 - 8x^3$
4. $250x^3 - 16y^3$
5. $5\sqrt{5}\,a^3 - 343$
6. $3\sqrt{3}\,x^3 - 5\sqrt{5}\,y^3$
7. $a^6 - 64$
8. $x^6 - y^6$
9. $8x^6 - y^3$
10. $x^{\frac{3}{2}} - y^{\frac{3}{2}}$
11. $ab^4 - 8a^4b$

### 4.4 Factorisation of $x^3 + y^3 + z^3 - 3xyz$

We notice that

$$x^3 + y^3 + z^3 - 3xyz$$

$$= x^3 + y^3 + 3xy(x + y) + z^3 - 3xyz - 3xy(x + y)$$

[adding and subtracting $3xy(x + y)$]

$$= (x + y)^3 + z^3 - 3xy\{z + (x + y)\}$$

$$= (x + y + z)\{(x + y)^2 - (x + y)z + z^2\} - 3xy(x + y + z)$$

[using formula 1]

$= (x + y + z)[x^2 + y^2 + 2xy - xz - yz + z^2 - 3xy]$
$= (x + y + z)[x^2 + y^2 + z^2 - xy - yz - zx]$

**Formula 3 :** $\mathbf{x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}$

Corollary : If $x + y + z = 0$ then $x^3 + y^3 + z^3 = 3xyz$

Clearly $x^3 + y^3 + z^3 - 3xyz$
$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$
$= 0 \quad$ if $x + y + z = 0$

Thus. $x^3 + y^3 + z^3 = 3xyz$ if $x + y + z = 0$

Now we solve some examples in the following using the above formula.

**Example 1 :** Factorise $8x^3 + 27y^3 + 125z^3 - 90xyz$

**Solution :** $8x^3 + 27y^3 + 125z^3 - 90xyz$
$= (2x)^3 + (3y)^3 + (5z)^3 - 3(2x)(3y)(5z)$
$= (2x + 3y + 5z)(4x^2 + 9y^2 + 25z^2 - 6xy - 10xz - 15yz)$

**Example 2 :** Find the factors of $27x^3 - y^3 + 8z^3 + 18xyz$

**Solution :** $27x^3 - y^3 + 8z^3 + 18xyz$
$= (3x)^3 + (-y)^3 + (2z)^3 - 3(3x)(-y)(2z)$
$= (3x - y + 2z)(9x^2 + y^2 + 4z^2 + 3xy - 6xz + 2yz)$

**Example 3 :** Factorise $a^6 - 8y^3 - 125z^3 - 30a^2yz$

**Solution :**

$a^6 - 8y^3 - 125z^3 - 30a^2yz$
$= (a^2)^3 + (-2y)^3 + (-5z)^3 - 3(a^2)(-2y)(-5z)$
$= (a^2 - 2y - 5z)(a^4 + 4y^2 + 25z^2 + 2a^2y + 5a^2z - 10yz)$
(by formula 3)

**Example 4 :** Find the factors of

$x^{\frac{3}{2}} - y^{\frac{3}{2}} + 8z^{\frac{3}{2}} + 6(xyz)^{\frac{1}{2}}$

**Solution :**

$x^{\frac{3}{2}} - y^{\frac{3}{2}} + 8z^{\frac{3}{2}} + 6(xyz)^{\frac{1}{2}}$
$= (x^{\frac{1}{2}})^3 + (-y^{\frac{1}{2}})^3 + (2z^{\frac{1}{2}})^3 - 3(x^{\frac{1}{2}})(-y^{\frac{1}{2}})(2z^{\frac{1}{2}})$
$= (x^{\frac{1}{2}} - y^{\frac{1}{2}} + 2z^{\frac{1}{2}})\{x + y + 4z + (xy)^{\frac{1}{2}} - 2(xz)^{\frac{1}{2}} + 2(yz)^{\frac{1}{2}}\}$

**Example 5 :** Show that $(b-c)^3 + (c-a)^3 + (a-b)^3 = 3(b-c)(c-a)(a-b)$

**Solution :** Let $b-c=x$, $c-a=y$ and $a-b=z$

Then $x + y + z = b - c + c - a + a - b = 0$

Hence, by corollary we get $x^3 + y^3 + z^3 = 3xyz$

Replacing $x, y, z$ by their respective values we get
$(b-c)^3 + (c-a)^3 + (a-b)^3 = 3(b-c)(c-a)(a-b)$

**Example 6 :** Show that $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3=3abc(a-b)(b-c)(c-a)$

**Solution :** Suppose that $a(b-c)=x$, $b(c-a)=y$ and $c(a-b)=z$

Then, $x+y+z=a(b-c)+b(c-a)+c(a-b)$
$=ab-ac+bc-ba+ca-cb$
$=0$

and so $x^3+y^3+z^3=3xyz$

or $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3=3a(b-c)\,b(c-a)\,c(a-b)$
$=3abc(b-c)(c-a)(a-b)$

## Exercises 4.3

1. Factorise the following :
   (i) $a^3-b^3+c^3+3abc$ (ii) $a^3-8b^3+125c^3+30abc$
   (iii) $8a^3-\dfrac{b^3}{8}-c^3-3abc$ (iv) $2\sqrt{2}\,a^3+b^3+c^3-3\sqrt{2}\,abc$
2. Find the factors in each of the following parts :
   (i) $7b^3+c^3+6b^2c$ (ii) $1+b^3+c^3-3bc$
   *Hint* : Write $7b^3=8b^3-b^3$
   (iii) $a^3-8+125c^3+30ac$ (iv) $2\sqrt{2}+1+c^3-3\sqrt{2}\,c$
3. Prove that
   $(b+c)^3(b-c)^3+(c-a)^3(c+a)^3+(a-b)^3(a+b)^3=3(a^2-b^2)(b^2-c^2)(c^2-a^2)$
4. Factorise :
   (i) $x^3+y^3+z^3$ if $x+y=-z$
   (ii) $x^3+y^3-8$ if $x-2=-y$
   (iii) $a^3+8b^3+1$ if $a+1=-2b$

### 4.5 Factorisation of $x^4-y^4$

We find that

$$x^4-y^4=(x^2)^2-(y^2)^2$$
$$=(x^2-y^2)(x^2+y^2) \text{ [since } a^2-b^2=(a-b)(a+b)]$$
$$=(x-y)(x+y)(x^2+y^2)$$

We can show that $x^2+y^2$ cannot be factorised into linear factors. Thus we have

**Formula 4 :** $x^4-y^4=(x-y)(x+y)(x^2+y^2)$

Now we use this formula in the following examples to illustrate its use for factorisation.

**Example 1 :** Factorise $a^4-16$

**Solution :** $a^4-16=a^4-2^4$
$=(a-2)(a+2)(a^2+4,$

**Example 2 :** Find the factors of $25x^4y^4 - z^4$

**Solution :** $25x^4y^4 - z^4$

$= (\sqrt{5}\,xy)^4 - z^4$

$= (\sqrt{5}\,xy - z)(\sqrt{5}\,xy + z)(5x^2y^2 + z^2)$

**Example 3 :** Factorise $3a - 243a^5$

**Solution :** $3a - 243a^5$

$= 3a(1 - 81a^4)$

$= 3a[1 - (3a)^4]$

$= 3a(1 - 3a)(1 + 3a)(1 + 9a^2)$ (by formula 4)

**Example 4 :** Find the factors of $x^{\frac{4}{3}} - 16y^{\frac{4}{3}}$

**Solution :** $x^{\frac{4}{3}} - 16y^{\frac{4}{3}}$

$= (x^{\frac{1}{3}})^4 - (2y^{\frac{1}{3}})^4$

$= (x^{\frac{1}{3}} - 2y^{\frac{1}{3}})(x^{\frac{1}{3}} + 2y^{\frac{1}{3}})(x^{\frac{2}{3}} + 4y^{\frac{2}{3}})$

**Example 5 :** Factorise $a^4 - 49b^4$

**Solution :** $a^4 - 49b^4$

$= a^4 - (\sqrt{7}\,b)^4$

$= (a - \sqrt{7}\,b)(a + \sqrt{7}\,b)(a^2 + 7b^2)$

## Exercises 4.4

1. Factorise the polynomials in each part :

   (i) $a^4 - 1$ (ii) $16a^4 - 625b^4$

   (iii) $a^2b^6 - a^6b^2$ (iv) $x^8 - y^8$

   (v) $x^{\frac{4}{5}} - y^{\frac{4}{5}}$ (vi) $x^{\frac{4}{3}} - 81y^{\frac{4}{3}}$

   (vii) $x^4 - 9y^4$

2. Show that $(a+1)^4 - a^4 = (2a+1)(2a^2+2a+1)$

3. Show that $(2x+1)^4 - (2x-1)^4 = 16x(4x^2+1)$

4. Show that the difference of the fourth powers of two consecutive even integers is divisible by 16 but not by 32.

(*Hint* : Two consecutive even integers are $2n$ and $2n+2$.)

5. Show that the difference of the fourth powers of two consecutive odd integers is divisible by 16.

(*Hint :* Two consecutive odd integers are $2n-1$ and $2n+1$.)

## H.C.F. AND L.C.M. OF TWO POLYNOMIALS

We have already learnt in earlier classes how to find the Highest Common Factor (H.C.F.) and the Least Common Multiple (L.C.M.) of two numbers. The concept of H.C.F. and L.C.M. can be extended to polynomials also. We will here study the H.C.F. and L.C.M. of only those polynomials which have integral coefficients.

We illustrate the concept of H.C.F. through some examples solved below.

**Example 1 :** Find the H.C.F. of $x^2$ and $x^5$.

**Solution :** We find the sets of factors of $x^2$ and $x^5$ first. The set of factors of $x^2$ is $\{1, x, x^2\}$ and the set of factors of $x^5$ is $\{1, x, x^2, x^3, x^4, x^5\}$.

Now we find the set of common factors, which is $\{1, x, x^2\}$ and this is the intersection of the sets of factors of $x^2$ and $x^5$.

Hence the H.C.F. of $x^2$ and $x^5$ is $x^2$, being the **common factor of highest degree.**

**Example 2 :** Find the H.C.F. of $(x-2)(x+2)$ and $(x+2)(x+3)$.

**Solution :** The set of factors of $(x-2)(x+2)$ is $\{1, (x-2), (x+2), (x-2)(x+2)\}$ and of $(x+2)(x+3)$ is $\{1, (x+2), (x+3), (x+2)(x+3)\}$.

Thus, the set of common factors is $\{1, (x+2)\}$.

We now notice that $(x+2)$ is the common factor of highest degree and hence is the H.C.F.

**Example 3 :** Find the H.C.F. of $x^2-3x+2$ and $x^2+6x-7$.

**Solution :** We factorise $x^2-3x+2$, and see that $x^2-3x+2=(x-1)(x-2)$. The set of factors of $x^2-3x+2$ is thus $\{1, (x-1), (x-2), (x-1)(x-2)\}$

In the same manner, we find that the set of factors of

$$x^2+6x-7 \text{ is } \{1, (x-1), (x+7), (x-1)(x+7)\}$$

The set of common factors is $\{1, (x-1)\}$ and thus the H.C.F. is $(x-1)$.

We may note that the H.C.F. of two non-zero polynomials $f(x)$ and $g(x)$ is a polynomial $h(x)$ with the following two properties :

(*i*) $h(x)$ is a factor of both $f(x)$ and $g(x)$, that is $h(x)$ is a common factor of $f(x)$ and $g(x)$.

(*ii*) The degree of $h(x)$ is highest among the common factors of $f(x)$ and $g(x)$.

It is a convention to take H.C.F. to be such that the co-efficient of the highest power in $x$ is positive.

The above two properties of the H.C.F. of two non-zero polynomials $f(x)$ and $g(x)$ lead us to the following procedure for finding their H.C.F.

1. Express the polynomials $f(x)$ and $g(x)$ in the complete factored form as a product of numerical factor, linear factors and quadratic factors if linear factors are not possible.
2. Find the H.C.F. of the numerical factors of $f(x)$ and $g(x)$.
3. Find the factors of highest power common to the two polynomials $f(x)$ and $g(x)$.
4. Find the product of all such highest power of common factors and the H.C.F. of the numerical factors. This is the H.C.F. of $f(x)$ and $g(x)$.

We solve a few more examples.

**Example 4 :** Find the H.C.F. of $6x^2 - 18x + 12$ and $8x^3 - 48x^2 + 88x - 48$

**Solution :** Expressing the two polynomials in the factored form we get

$$6x^2-18x+12=6(x-1)(x-2) \text{ and } 8x^3-48x^2+88x-48=8(x-1)(x-2)(x-3)$$

The H.C.F. of the numerical factors 6 and 8 is 2. The highest power common factors are $(x-1)$ and $(x-2)$. Hence the H.C.F. is $2(x-1)\,(x-2)$.

**Example 5 :** Find the H.C.F. of $x^3+8x^2+14x+6$ and $x^3+x^2-x-1$.

**Solution :** Factorising the two polynomials, we get

$$x^3 + 8x^2 + 14x + 6 = (x + 1)^2\,(x + 6)$$

and $x^3 + x^2 - x - 1 = (x + 1)^2\,(x-1)$. The highest power common factor is $(x+1)^2$. There are no other common factors and hence the H.C.F. is $(x + 1)^2$.

We have learnt how to find H.C.F. of two polynomials through the process of factorisation. The H.C.F. of two numbers can be found by a process of successive division also. This process may also be used for finding the H C.F. of two polynomials.

It sometimes proves to be simpler than the earlier process. We shall now solve some examples to illustrate this process.

**Example 6 :** Find the H.C.F. of $x^2 - 5x + 6$ and $x^3 - 2x^2 + x - 2$

**Solution :** We divide one polynomial by the other. Here we divide $x^3-2x^2+x-2$ by $x^2-5x+6$ because $x^3 - 2x^2 + x - 2$ is of higher degree.

$$\begin{array}{r|l}
 & x + 3 \\
\hline
x^2-5x+6 & x^3 - 2x^2 + x - 2 \\
 & x^3 - 5x^2 + 6x \\
\hline
 & 3x^2 - 5x - 2 \\
 & 3x^2 - 15x + 18 \\
\hline
 & 10x - 20
\end{array}$$

The remainder is $10x - 20$. To make the process simpler we divide the remainder by numerical factor 10, which is common to the two terms in the remainder. We get $x - 2$, which now becomes the new divisor and the earlier divisor $x^2 - 5x + 6$ is new dividend.

We then have

$$\begin{array}{r|l}
 & x - 3 \\
\hline
x - 2 & x^2 - 5x + 6 \\
 & x^2 - 2x \\
\hline
 & -3x + 6 \\
 & -3x + 6 \\
\hline
 & 0
\end{array}$$

Since the remainder is zero now, the process stops. The last divisor $(x - 2)$ is the H.C.F. of the two polynomials. We may try to find the H.C.F. by the earlier method also and we will see that it is the same.

**Example 7 :** Find the H.C.F. of the polynomials

$$x^4 + x^3 + 3x^2 + 2x + 2 \text{ and } 2x^3 - x^2 + 4x - 2$$

**Solution :** We divide $x^4 + x^3 + 3x^2 + 2x + 2$ by $2x^3 - x^2 + 4x - 2$. Why ?

$$
\begin{array}{r|l}
 & \frac{x}{2} + \frac{3}{4} \\
2x^3 - x^2 + 4x - 2 & x^4 + x^3 + 3x^2 + 2x + 2 \\
 & x^4 - \frac{x^3}{2} + 2x^2 - x \\
\hline
 & \frac{3x^3}{2} + x^2 + 3x + 2 \\
 & \frac{3x^3}{2} - \frac{3x^2}{4} + 3x - \frac{3}{2} \\
\hline
 & \frac{7x^2}{4} + \frac{7}{2}
\end{array}
$$

The remainder is $\frac{7x^2}{4} + \frac{7}{2} = \frac{7}{4}(x^2 + 2)$. We remove the numerical factor $\frac{7}{4}$ to avoid fractions in the next division, so the new divisor is $x^2 + 2$ and the new dividend is $2x^3 - x^2 + 4x - 2$. We start the division process again.

$$
\begin{array}{r|l}
 & 2x - 1 \\
x^2 + 2 & 2x^3 - x^2 + 4x - 2 \\
 & 2x^3 + 4x \\
\hline
 & -x^2 - 2 \\
 & -x^2 - 2 \\
\hline
 & 0
\end{array}
$$

Since the remainder is zero now, the process stops. The last divisor is $x^2 + 2$. Hence the H.C.F. is $x^2 + 2$. It may be noted that this process gives H.C.F. more easily in this example, since it is not easy to factorise $x^4 + x^3 + 3x^2 + 2x + 2$.

Allied to the concept of H.C.F. of two polynomials is the concept of Least Common Multiple (L.C.M.) of two polynomials. Just as the method of finding H.C.F. of two polynomials follows the pattern of finding the H.C.F. of two numbers, similarly the method of finding the L.C.M. of two polynomials follows the pattern of finding the L.C.M. of two numbers.

We will now learn how to find the L.C.M. of two polynomials through some solved examples below.

**Example 8 :** Find the L.C.M. of $4x^3$ and $6x^4$.

**Solution :** We notice that the L.C.M. of the constant factors 4 and 6 is 12. Also the L.C.M. of $x^3$ and $x^4$ is $x^4$. Using the method of finding the L.C.M. of two numbers we see that the L.C.M. of $4x^3$ and $6x^4$ is $12x^4$.

**Example 9 :** Find the L.C.M. of $2x^2 - 8$ and $x^2 - 5x + 6$.

**Solution :** First, we factorise the two polynomials.

We find that $2x^2 - 8 = 2(x - 2)(x + 2)$

and $x^2 - 5x + 6 = (x - 2)(x - 3)$

Now using the method of finding the L.C.M. of two numbers we find the required L.C.M. to be $2(x-2)(x+2)(x-3)$.

**Example 10 :** Find the L.C.M. of the polynomials

$$2(x^3 + x^2 - x - 1) \text{ and } 3(x^3 + 5x^2 + 7x + 3)$$

**Solution :** First, we factorise the two polynomials.

We find that $2(x^3 + x^2 - x - 1) = 2(x+1)^2(x-1)$

and $3(x^3 + 5x^2 + 7x + 3) = 3(x+1)^2(x+3)$

Following the method of examples 8 and 9 we find that the required L.C.M. is

$$6(x+1)^2(x+3)(x-1).$$

Thus we notice that the L.C.M. of two polynomials $f(x)$ and $g(x)$ is a polynomial $l(x)$ with the following two properties :

(*i*) $l(x)$ is a multiple of both $f(x)$ and $g(x)$.

(*ii*) The degree of $l(x)$ is the least among the common multiples.

It is also a convention to take the L.C.M. to be such that the coefficient of the highest power of $x$ in it is positive.

We notice that the definition of L.C.M. of two polynomials has some kind of similarity to the definition of their H.C.F. Thus, the procedure for finding the L.C.M. of the two polynomials will have some similarity to the procedure for finding the H.C.F.

We give below the procedure for finding the L.C.M. of two polynomials.

1. Write each of the polynomials in the complete factored form, *i.e.*, as a product of numerical factor, linear factors and quadratic factors if linear factors are not possible.
2. The L.C.M. is the product of the following :
   (*i*) L.C.M. of the numerical factors in the two polynomials.
   (*ii*) The highest power, common factors of the two polynomials.
   (*iii*) The factors of highest power, other than the common factors, present in the two polynomials.

As we can see, we have used the above procedure for finding the L.C.M. of two polynomials. However, we have another method of finding the L.C.M. of two polynomials at our disposal, similar to the method followed in the case of numbers.

We have

$$\text{L.C.M. of two polynomials} = \frac{\text{Product of the two polynomials}}{\text{H.C.F. of the two polynomials}}$$

This method of finding the L.C.M. is particularly useful in cases where it is difficult to find factors of the two polynomials. We can find the H.C.F. by the division process. The L.C.M. can then be found by the above formula.

Let us verify the above formula in examples 8 and 9 solved earlier. In example 8, we see that the H.C.F. is $2x^2$.

According to the formula for L.C.M., we have

$$\text{L.C.M.} = \frac{4x^3 \cdot 6x^4}{2x^3} = 12x^4$$

which is the same as found earlier.

In example 9, the polynomials are $2x^2 - 8$ and $x^2 - 5x + 6$.

We find their H.C.F. to be $(x - 2)$. According to the formula,

$$\text{L.C.M.} = \frac{(2x^2 - 8)(x^2 - 5x + 6)}{x - 2} = \frac{2(x^2 - 4)(x^2 - 5x + 6)}{x - 2}$$

$$= \frac{2(x-2)(x+2)(x-2)(x-3)}{x - 2}$$

$$= 2(x - 2)(x + 2)(x - 3)$$

which is the same as found earlier.

Just as we have found H.C.F. and L.C.M. of polynomials in $x$, similarly we can find H.C.F. and L.C.M. of polynomials in $x$ and $y$.

We will now slove an example to find the H.C.F. and L.C.M. of two polynomials in $x$ and $y$.

**Example 11 :** Find the L.C.M. of $x^3 + 10x^2y + 27xy^2 + 18y^3$ and $x^3 + 4x^2y + 3xy^2$

**Solution :** $x^3 + 10x^2y + 27xy^2 + 18y^3 = (x + y)(x + 3y)(x + 6y)$

$x^3 + 4x^2y + 3xy^2 = x(x + y)(x + 3y)$

So L.C.M. $= x(x + y)(x + 3y)(x + 6y)$

## Exercises 4.5

1. Find the H.C.F., by factorisation method, in each of the following parts :
   (*i*) $x^2 - 9, \quad x^2 - 6x + 9$
   (*ii*) $4x^2 - 16, 6x^2 + 24x + 24$
   (*iii*) $x^2 - 7x + 12, x^2 - 16$
   (*iv*) $x^3 - 3x^2 - 9x + 27, x^2 - 6x + 1$
   (*v*) $4x^3 - 16x^2 + 20x - 8, 8x^3 - 48x^2 + 72x - 32$
   (*vi*) $x^3 + x^2 + x + 1, x^4 - 1$
2. Find the H.C.F. of polynomials, in each of the following parts, using successive division :
   (*i*) $x^2 - 9 ; x^2 - 6x + 9$

(ii) $20x^2 - 9x + 1$ ; $5x^2 - 6x + 1$

(iii) $x^3 - 9x^2 + 23x - 15$ ; $4x^2 - 16x + 12$

(iv) $2x^3 + 2x^2 + 2x + 2$ ; $6x^3 + 12x^2 + 6x + 12$

(v) $x^3 - 3x^2 + 4x - 12$ ; $x^4 + x^3 + 4x^2 + 4x$

3. Find the L.C.M. of the polynomials in each of the following parts :

(i) $2x^2 - 10x + 12$ ; $x^2 - 6x + 5$

(ii) $x^2 - 1$ ; $(x-1)^2$

(iii) $x^2 + 2x + 1$ ; $5x^3 - 15x + 10$

(iv) $x^3 - 12x^2 + 44x - 48$ ; $x^2 - 10x + 24$

(v) $x^3 - 9x^2 + 23x - 15$ ; $4x^2 - 16x + 12$

(vi) $x^3 + x^2 + x + 1$ ; $2x^4 - 2$

(vii) $x^4 - y^4$ ; $x^3 + 2x^2y - xy^2 - 2y^3$

(viii) $x^3 - 4x^2y + 5xy^2 - 2y^3$ ; $x^3 - 5x^2y + 8xy^2 - 4y^3$

(ix) $x^3 - 10x^2 + 31x - 30$ ; $2x^3 - 8x^2 + 2x + 12$

(x) $x^3 - x^2 - 25x - 30$ ; $x^3 + 4x^2 - 5$

CHAPTER 5

# Shares and Dividend

## 5.1 Introduction

Suppose a new factory is to be set up for the manufacture of automobiles and it is estimated that a sum of Rs 1, 00,00,000 (one crore) is required to carry out the project. As it is not always possible for a single person to invest such a huge amount, some persons interested in the project form a company called a Joint Stock Company. They divide the required capital into small portions called shares which may be of any value from Rs 10 (or less) upto Rs 100 (or more). Each person who purchases one or more shares is called a shareholder and is a part owner of the company. Each shareholder gets a share certificate indicating the number of shares he holds in the company. Thus, a company's capital is divided into equal portions called shares which are bought by persons, called the shareholders.

When the factory starts selling automobiles, it earns money. Part of the income is used in paying working expenses, and the remainder is termed as profit. Each shareholder is entitled to a portion of the company's profit corresponding to the number of shares he holds.

## 5.2 Shares, Dividend and Debentures

The company (joint stock company) so formed and registered with the government is called a limited company. It can issue two types of shares, namely, preference shares and equity shares or ordinary shares.

### Preference Shares

Preference shares are those which carry preference both regarding dividend and the return of capital over equity shares. The rate of dividend on a preference share is fixed at the time of issue. Before any dividend is paid on other shares, dividend on preference shares at the fixed rate must be paid.

### Equity Shares

Shares which are not preference shares are known as equity shares or ordinary shares. The dividend on these shares is paid after the dividend on preference shares has been paid. The rate of dividend on these shares depends upon the amount of profit available.

In case the company goes into liquidation, *i.e.* the company is wound up, the amount of equity capital will be repayable only after every other claim including that of preference shareholders has been settled.

**Deferred Shares**

In some cases a company also issues shares called deferred shares. The holders of such shares receive no dividend unless dividend on other shares (preference and ordinary) has been paid at a certain rate or rates.

**Debentures**

A debenture is an acknowledgement of a debt of a company. Debentures are equal parts of a loan raised by a company. A debenture holder is a creditor of the company and is not a part owner. He is entitled to a fixed return every year irrespective of profits. His claim ranks prior to preference and equity shareholders. In the event of winding up of the company the amount due to debenture holders must be paid before anything is paid to the shareholders.

**Stock**

Sometimes a company issues a certificate called the stock certificate instead of share certificate indicating that a person holds a stock of, say, Rs 5000, *i.e.* shares worth Rs 5000.

**Dividend**

When the company makes a profit, part of that profit is divided amongst the shareholders and it is called the dividend.

**Market Value**

The value of a share mentioned in the share certificate is called its nominal value, face value or par value.

If a shareholder needs his money, he cannot claim it back from the company. He can, however, sell his share certificate/certificates directly to a buyer or in the market through a broker. The value of the share quoted in the market is called the Market Value.

The market value of the share varies from time to time depending upon the performance (profit) of a company, so that the price at which a man sells his shares may not be the same at which he bought them.

If the market value of a share is the same as the par value (face value) then the share is said to be at par.

If the market value is less than the face value, then it is said to be at a discount or below par. If it is more than the face value, then it is said to be at a premium or above par.

*Note* :

1. The market value of a share can be of any amount irrespective of its face value.
2. Face value of a share usually is Rs 10 or Rs 100 these days.
3. Dividend is always calculated on the face value of a share and is generally expressed as percentage.

In the following we work out a few examples to illustrate the concepts explained in the preceding pages.

**Example 1 :** A company declares an annual dividend of 8%. Find the annual income of a shareholder owning 30 shares at par value of Rs 100.

**Solution :** Annual dividend per share

$$= 8\% \text{ of Rs } 100$$

$$= \text{Rs } \frac{8}{100} \times 100 = \text{Rs } 8$$

$\therefore$ Annual dividend from 30 shares

$$= \text{Rs } 8 \times 30$$

$$= \text{Rs } 240$$

**Example 2 :** Find the annual dividend on 500 shares of a stock with par value of Rs 10 if the quarterly dividend is 6% of the par value.

**Solution :** Quarterly dividend per share

$$= 6\% \text{ of Rs } 10$$

$$= \text{Rs } \frac{6}{100} \times 10 = \text{Rs } \frac{6}{10}$$

Annual dividend per share

$$= \text{Rs } \frac{6}{10} \times 4$$

$$= \text{Rs } \frac{12}{5}$$

$\therefore$ Annual dividend on 500 shares

$$= \text{Rs } \frac{12}{5} \times 500$$

$$= \text{Rs } 1200$$

**Example 3 :** A man holds 15 debentures of a company and receives a dividend of Rs 18.75 per quarter. If the dividend he receives be 8% per annum, find the value of a debenture.

**Solution :** Annual dividend of 15 debentures

$$= \text{Rs } 18.75 \times 4$$

$$= \text{Rs } 75$$

Suppose the value of a debenture = Rs $x$

Value of 15 debentures = Rs $15x$

Annual dividend $= \text{Rs } 15\,x \times \frac{8}{100}$

$$= \text{Rs } \frac{6x}{5}$$

$\therefore$ $$\frac{6x}{5} = 75$$

*i.e.* $$x = \frac{75 \times 5}{6}$$

$$= 62.50$$

$\therefore$ Value of a debenture = Rs 62.50

**Example 4** : Find the cost of 40 cloth-mill shares of Rs 25 each quoted at Rs 35. Find also the gain to the original shareholder.

**Solution** : Original cost of 1 share = Rs 25

Original cost of 40 shares = Rs 25 $\times$ 40 = Rs 1000

Cost (Market Value) of 1 share = Rs 35

Cost (Market Value) of 40 shares = Rs 35 $\times$ 40 = Rs 1400

Gain to the shareholder = Rs 1400 − Rs 1000 = Rs 400

**Example 5** : What is gained by buying 500 shares (par value Rs 100) at 10% discount and selling them again at 5% premium ?

**Solution** : Par value of a share = Rs 100

Market price of a share

= Rs 100 — Rs 10

= Rs 90

Amount invested in buying

500 shares = Rs 500 $\times$ 90

= Rs 45000

Market price of a share while it is being sold

= Rs 100 + Rs 5

= Rs 105

Amount obtained on selling 500 shares

= Rs 500 $\times$ 105

= Rs 52500

$\therefore$ Gain = Rs 52500 − Rs 45000

= Rs 7500

**Example 6** : What income is derived by investing Rs 2910 in 11% stock at Rs 97? (*The phrase 11% stock at Rs 97 means the market value of a stock of Rs 100 is Rs 97 and the owner gets a dividend of Rs 11 on every Rs 100 stock.*)

**Solution** : Amount of stock bought for Rs 97

= Rs 100

Amount of stock bought for Rs 2910

$$= \text{Rs } \frac{100}{97} \times 2910$$

= Rs 3000

$$\text{Income from Rs } 100 \text{ stock} = \text{Rs } 11$$

$$\therefore \quad \text{Income from Rs } 3000 \text{ stock} = \text{Rs } \frac{11}{100} \times 3000$$

$$= \text{Rs } 330$$

## Exercises 5.1

1. Find the annual dividend on 450 shares of a stock with par value of Rs 10 if the annual dividend is 7.5%.
2. Find the amount of dividend received annually by a stock holder, having 150 shares with par value of Rs 50 if the half yearly dividend is 5%.
3. A man holds 20 shares of a bank and receives a dividend of Rs 15 per quarter. If the dividend he receives be 12% per annum, find the value of a share.
4. Find the cost of 200 shares of Rs 10 each available at Rs 2.50 premium.
5. Find the cost of 220 shares of Rs 15 each available at Rs 1.50 discount.
6. What is the cost of purchasing Rs 2000 stock yielding 9% at 105 ?
7. What income is derived by investing Rs 3480 in 8% stock at 87 ?
8. Find the market price of a 9% stock when the investment of Rs 1800 yields a dividend of Rs 135.
9. A company declared a dividend of 6% semi-annually. Find the annual income of a shareholder owning 135 shares at par value of Rs 10.
10. The capital stock of a Cement Company is Rs 3,00,000 and is divided into 3000 equity shares. If the company pays a dividend of Rs 56000, what amount will a person receive for 36 shares ?
11. A man buys 20 shares (par value Rs 10) of a company which pays 9% dividend, at such a price that he gets 12% on his money. Find the market value of a share.
12. Naresh holds 25 shares of a company and receives Rs 137.50 half-yearly as dividend. If the dividend he receives be 11% per annum, find the value of a share.
13. A man invests Rs 6000 in a company paying 8% per annum as dividend when its Rs 80 share is selling for Rs 120. What is his annual income and what percentage does he obtain on his investment ?
14. A man sells 2000 ordinary shares (par value Rs 10) of a tea company, which pays a dividend of 25% at Rs 33.00 per share. He invests the proceeds in Cotton Textiles (par value Rs 25.00) ordinary shares at Rs 44 per share, which pays a dividend of 15%. Find (*i*) the number of Cotton Textiles shares purchased and (*ii*) change in his dividend income.

15. A man invests a certain sum in 7% debentures at Rs 104. When the price falls to Rs 101, he sells out and loses Rs 600 in the transaction. Find the sum invested.

## 5.3 Brokerage

The stock is generally bought or sold through a broker who charges a small commission called the Brokerage. The brokerage is always calculated on the face value.

A buyer has to pay the market value together with the brokerage and a seller gets market value reduced by the brokerage *i.e.*,

Amount paid by the buyer = Market Value + Brokerage

Amount received by the seller = Market Value − Brokerage

In the preceding pages, we have studied some examples on shares and dividend which were quite simple. It is proposed to study in the following, some examples which include the concept of brokerage and also some others in which incomes derived from different investments are compared.

**Example 1** : Which is better investment, 12% stock for Rs 108 or 8% stock for Rs 90 ?

**Solution** : Suppose the total investment

$$= \text{Rs } 108 \times 90$$

Total stock in the first case

$$= \text{Rs } \frac{100}{108} \times 108 \times 90$$

$$= \text{Rs } 9000$$

$$\text{Dividend} = \text{Rs } 9000 \times \frac{12}{100}$$

$$= \text{Rs } 1080$$

Total stock in the second case

$$= \text{Rs } \frac{100}{90} \times 108 \times 90$$

$$= \text{Rs } 10800$$

$$\text{Dividend} = \text{Rs } 10800 \times \frac{8}{100}$$

$$= \text{Rs } 864$$

$\therefore$ First investment is better.

**Example 2** : A man wishes to invest Rs 2500. He invests Rs 900 in 8% stock at Rs 75, Rs 1000 in 7% stock at Rs 70 and the remainder in 9% stock. If the total yield from his investment is 10%, at what price did he buy the 9% stock ?

**Solution** : Amount of stock in the first case

$$= \text{Rs } \frac{100}{75} \times 900$$

$$= \text{Rs } 1200$$

$$\text{Dividend} = \text{Rs } 1200 \times \frac{8}{100}$$
$$= \text{Rs } 96$$

Amount of stock in the second case

$$= \text{Rs } \frac{100}{70} \times 1000$$
$$= \text{Rs } \frac{10000}{7}$$
$$\text{Dividend} = \text{Rs } \frac{10000}{7} \times \frac{7}{100}$$
$$= \text{Rs } 100$$

Investment in the third case

$$= \text{Rs } 2500 - \text{Rs } 900 - \text{Rs } 1000$$
$$= \text{Rs } 600$$

Suppose he invests Rs 600 in 9% stock at Rs $X$

$$\therefore \quad \text{Stock in the third case} = \text{Rs } \frac{100}{X} \times 600$$
$$= \text{Rs } \frac{60000}{X}$$
$$\text{Dividend} = \text{Rs } \frac{60000}{X} \times \frac{9}{100}$$
$$= \text{Rs } \frac{5400}{X}$$
$$10\% \text{ of the total investment} = \text{Rs } 2500 \times \frac{10}{100}$$
$$= \text{Rs } 250$$
$$\therefore \quad 96 + 100 + \frac{5400}{X} = 250$$
$$\therefore \quad X = 100$$

$\therefore$ He invests Rs 600 in 9% stock at Rs 100.

**Example 3 :** A company has a capital stock of Rs 2,00,000 divided into 500 shares of 6% preferred stock and 1500 shares of ordinary stock, each with par value of Rs 100 per share. The company declares a dividend of Rs 15500. If a man holds 30 shares of preferred stock and 75 shares of ordinary stock, find the amount of dividend he receives.

**Solution :** Total declared dividend = Rs 15500

Dividend of preferred stock per share

$$= 6\% \text{ of Rs } 100$$
$$= \text{Rs } 6$$

Dividend of preferred stock from 500 shares

$$= \text{Rs } 6 \times 500 = \text{Rs } 3000$$

Remaining dividend $=$ Rs 15500—Rs 3000
$=$ Rs 12500

$\therefore$ Dividend of common stock per share

$= \text{Rs } \frac{12500}{1500}$

$= \text{Rs } \frac{25}{3}$

Dividend from 30 shares of preferred stock

$= \text{Rs } 6 \times 30$

$= \text{Rs } 180$

Dividend from 75 shares of ordinary stock

$= \text{Rs } \frac{25}{3} \times 75$

$= \text{Rs } 625$

Total dividend $= \text{Rs } 180 + \text{Rs } 625$

$= \text{Rs } 805$

**Example 4 :** A man sells out Rs 5000 of a 9% stock at Rs 106 and invests the proceeds in a 13% stock at Rs 139. Find the amount of the new stock that he buys and the change in income. (Assume that brokerage is to be paid at the rate of 1% of the face value on each transaction.)

**Solution :** Brokerage $= 1\%$

Sale price of the stock of Rs 100 $=$ Rs 106 $-$ Re 1

$= \text{Rs } 105$

Sale price of Rs 5000 stock $= \text{Rs } 5000 \times \frac{105}{100}$

$= \text{Rs } 5250$

Purchase price of a stock of Rs 100

$= \text{Rs } 139 + \text{Re } 1$

$= \text{Rs } 140$

Amount of new stock bought $= \text{Rs } \frac{100}{140} \times 5250$

$= \text{Rs } 3750$

Income from the first stock $= \text{Rs } 5000 \times \frac{9}{100}$

$= \text{Rs } 450$

Income from the new stock $= \text{Rs } 3750 \times \frac{13}{100}$

$= \text{Rs } \frac{975}{2}$

$= \text{Rs } 487.50$

Change (increase) in income $=$ Rs 487.50 $-$ Rs 450

$= \text{Rs } 37.50$

**Example 5 :** How much cash must I pay for 500 shares of par value Rs 10 at a premium of Rs 2, brokerage $\frac{1}{2}$% of the face value. If the dividend is 10% per annum, find the half yearly dividend on these shares. What rate per cent per annum do I get for my money ?

**Solution :** Cost of one share = Rs 10 + Rs 2 + Rs 0.05
= Rs 12.05

Cost of 500 shares = Rs 12.05 × 500
= Rs 6025

Annual dividend on 500 shares

$$= \text{Rs } 5000 \times \frac{10}{100}$$

$$= \text{Rs } 500$$

Half yearly dividend = Rs 250

Annual interest on Rs 6025 = Rs 500

$$\text{Annual interest on Rs } 100 = \text{Rs } \frac{500}{6025} \times 100$$

$$= \text{Rs } 8.3 \text{ nearly}$$

∴ I get 8.3% per annum.

**Example 6 :** By investing Rs 8000 partly in 12% stock at Rs 80 and partly in 10% stock at Rs 120, I get an annual income of Rs 800. How much do I invest in each kind of stock ?

**Solution :** Suppose I invest Rs $X$ in 12% stock at Rs 80 and Rs $(8000 - X)$ in 10% stock at Rs 120.

Annual income from the first stock

$$= \text{Rs } X \times \frac{12}{80}$$

$$= \text{Rs } \frac{3X}{20}$$

Annual income from the second stock

$$= \text{Rs } (8000 - X) \times \frac{10}{120}$$

$$= \text{Rs } \frac{8000 - X}{12}$$

$$\text{Total annual income} = \text{Rs } \left(\frac{3X}{20} + \frac{8000 - X}{12}\right)$$

$$\therefore \quad \frac{3X}{20} + \frac{8000 - X}{12} = 800$$

$$9X + 40000 - 5X = 48000$$

$$4X = 8000$$

$$X = 2000$$

∴ My investment in the first stock = Rs 2000

**My investment in the second stock = Rs 6000**

**Example 7** : A man invests Rs 8370 partly in 9% stock at Rs 96 and the remainder in 12% stock at Rs 120. His dividend from each investment is the same. Find the amount invested in each kind of stock.

**Solution** : Suppose he invests Rs $X$ in 9% stock at Rs 96 and Rs $(8370 - X)$ in 12% stock at Rs 120.

Dividend from the first investment

$$= \text{Rs } X \times \frac{9}{96}$$

$$= \text{Rs } \frac{3X}{32}$$

Dividend from the second investment

$$= \text{Rs } (8370 - X) \times \frac{12}{120}$$

$$= \text{Rs } \frac{8370 - X}{10}$$

$$\therefore \quad \frac{3X}{32} = \frac{8370 - X}{10}$$

$$30X = 267840 - 32X$$

$$62X = 267840$$

$$X = 4320$$

$\therefore$ Amount invested in the first case

$$= \text{Rs } 4320$$

Amount invested in the second case

$$= \text{Rs } 8370 - \text{Rs } 4320$$

$$= \text{Rs } 4050$$

## Exercises 5.2

Which is the better investment :

1. 8% stock at 102 or 10% stock at 124 ?
2. 7% stock at 130 or $6\frac{3}{4}$% stock at 125 ?
3. A 8% stock is at 120 and a 9% stock is at 122. Which is the better investment ? What equal investments should be made in the two stocks to earn an annual income of Rs 514 ?
4. Find the change in income, when a person sells out Rs 5000 of a 6% stock at Rs 97 and invests the proceeds in 12% stock at Rs 119. (Brokerage 1% of the face value)
5. A man invests Rs 34000 partly in 8% stock at Rs 80 and partly in $7\frac{1}{2}$% stock at Rs 90. If his yearly income be Rs 3000, how much stock of each kind does he hold?

6. Equal sums of money are invested in $9\frac{1}{2}\%$ stock at 85 and $12\frac{1}{2}\%$ stock at 117. If one of these investments yields Rs 20 more than the other, what is the capital invested ? (brokerage $\frac{1}{2}\%$)
7. A man invested Rs 9000 partly in a 12% stock at Rs 96 and partly in a 16% stock at Rs 120 in such a manner that he gets Rs 1175. How much has he invested in each stock ?
8. A man invested Rs 1196 in 10% stock at 92 and sold out the stock when the price rose to 94. What did he gain ?
9. A man invested Rs 9000 in 6% stock at Rs 90. He sold the stock when the price rose to Rs 95 and invested the sale proceeds in 8% stock. By doing so his income increased by Rs 160. At what price did he buy the latter stock ?
10. A person invests Rs 3600 in the 9% stock at 89, and when it has risen to 94 he sells out and re-invests in the 14% stock at 111. Find the change in his income, 1% brokerage being charged on each transaction.
11. The capital stock of a company is Rs 65,00,000 and is divided into 2500 shares of 6% preferred stock and 62500 of ordinary stock. Both the preferred and the ordinary shares have a par value of Rs 100 per share. The net profit, in a given year, of the company is Rs 425000 out of which Rs 250000 is distributed as dividend. What amount will a man receive for his 25 shares of preferred stock and 85 shares of ordinary stock ?
12. A company has a capital stock of Rs 3,00,000 divided into 800 shares of 8% preferred stock with par value of Rs 100 per share and remaining shares of ordinary stock. The company declares a dividend of Rs 21800. If a man holds 40 shares of preferred stock and 60 shares of ordinary stock, find the amount of dividend he receives.
13. I invest half of a certain sum in a 8% stock at par and the other half in 10% stock at Rs 120. If I had invested one quarter of the sum in the first stock and the remainder in the second stock, I should have had Rs 10 more of annual income than in the former case. What total sum do I invest ?
14. The present income of a company would justify a dividend of 6% if there were no preference shares, but as Rs 50000 of the stock consists of such shares which are guaranteed $7\frac{1}{2}\%$ per annum, the ordinary shareholder gets only 5%. Find the amount of the ordinary stock of the company.
15. A person having invested Rs 800 in a 9% stock at Rs 75, sold it at a certain price and invested the proceeds in 13% debentures at Rs 96 and, thus, increased his income by Rs 8. At what price did he sell ?

CHAPTER 6

# Instalment Schemes

## 6.1 Instalment Purchase Schemes

People with small resources cannot buy a thing as and when they wish, but they may still need it badly. So, in order to facilitate the purchase of goods like bicycles, radios, sewing machines, fans, coolers, T.V. sets, cars, refrigerators, etc. a system known as instalment system/scheme is introduced by businessmen. In Instalment Purchase System the price quoted by the seller is higher than the price quoted for sale on cash payment basis. In other words under this instalment system, the buyer has to pay more than the actual price because of interest for deferred payments. Under this system the customer is not required to make full payment for the commodity at the time of purchase but is allowed to pay the same in easy monthly, quarterly, half-yearly, yearly instalments.

**Cash price** is the amount for which the article can be purchased on full payment.

**Cash down payment** is the payment made by the buyer at the time of signing the contract.

Instalment Scheme is of two types, viz. Hire Purchase Scheme and Instalment Purchase Scheme. In the former, goods are delivered to a person (hirer) who agrees to pay the owner (seller) equal/unequal instalments. Instalments are treated as hire charges for the goods until a certain fixed amount has been paid. In this case the legal ownership of the goods vests with the seller. The ownership of the goods is transferred to the hirer on the payment of the last instalment. The seller has the right to take away the goods, even if the last instalment remains unpaid. In the instalment purchase scheme, the buyer is not only in possession of the goods but also becomes the owner of the goods immediately after the contract of the sale is completed.

Instalment purchase scheme enables a person to buy goods which he would not be able to buy otherwise. Valuable articles are available to purchasers on convenient terms of payment. This method tends to encourage thrift. The buyer has to economise in future so that he may be able to pay his instalments for the convenience availed due to early possession of the goods.

Price under the Instalment Purchase System must not be confused with cash retail price (or cash price). Cash price is arrived at by the seller by adding profit to the cost price of the goods, while in the case of an Instalment Purchase Scheme, purchase price is arrived at by adding to the cost price, the profit and the interest on the cash price of the goods. Just after agreement, some initial payment which is called cash down payment

is to be made and the rest is paid in instalments which may be monthly, quarterly, half-yearly, yearly, etc. The shopkeeper charges interest on the amount arrived at, by deducting from the actual selling price the initial payment made at the time of purchase. This amount is considered as loan advanced to the customer by the shopkeeper.

In the following we work out a few examples to illustrate the concepts given above.

**Example 1 :** A watch is available for Rs 250 cash or for Rs 100 cash down payment followed by Rs 165 after six months. Find the rate of interest charged under instalment plan.

**Solution :** The watch is available for Rs 250 cash.

In the instalment plan, the cash down payment

= Rs 100

$\therefore$ The price to be paid in instalment has its present value

= Rs 250 − Rs 100 = Rs 150

Instalment = Rs 165

Let the rate of interest charged in the instalment plan be $r\%$ per annum.

Rs 150 would amount to Rs 165 at the end of 6 months.

$$\therefore \quad 150 + \frac{150 \times r \times 6}{100 \times 12} = 165$$

$$i.e., \quad 150 + \frac{3r}{4} = 165$$

Solving this equation for $r$, we get :

$$r = 20$$

$\therefore$ Rate = 20% per annum

**Example 2 :** A coat is sold for Rs 600 cash or for Rs 200 cash down payment followed by 2 monthly instalments of Rs 220 each. Find the rate of interest charged.

**Solution :** Cash price of the coat = Rs 600

In the instalment plan, cash down payment is Rs 200.

$\therefore$ Price to be paid in instalments has its present value equal to Rs (600 − 200), *i.e.* Rs 400.

Let the rate of interest charged under instalment plan be $r\%$ per annum.

$\therefore$ Rs 400 will amount to Rs $\left(400 + \frac{400 \times r \times 2}{100 \times 12}\right)$ ...(*i*)

at the end of 2nd month.

On the other hand at the end of 2nd month :

1st instalment of Rs 220 will amount to : Rs $\left(220 + \frac{220 \times r \times 1}{100 \times 12}\right)$

2nd instalment of Rs 220 will amount to : Rs $\left(220 + \frac{220 \times r \times 0}{100 \times 12}\right)$

= Rs 220

The two instalments taken together will amount to

$$\text{Rs}\left(440 + \frac{220\,r}{1200}\right) \quad ..(ii)$$

Thus, from (*i*) and (*ii*), we get :

$$400 + \frac{800\,r}{1200} = 440 + \frac{220\,r}{1200}$$

*i.e.* $$\frac{800\,r}{1200} - \frac{220\,r}{1200} = 440 - 400$$

*i.e.* $$\frac{580\,r}{1200} = 40$$

*i.e.* $$r = 40 \times \frac{1200}{580} = 82.75 \text{ (approx.)}$$

$\therefore$ Rate = 82.75% (approx.)

**Example 3** : An electric iron is sold either for Rs 150 cash or for Rs 36 cash down payment followed by Rs 25 a month for 5 months. Determine the rate of interest charged under instalment plan.

**Solution** : Cash price of the electric iron = Rs 150

In the instalment plan, the cash down payment is Rs 36.

$\therefore$ Price to be paid in instalments has its present value equal to Rs (150 − 36), *i.e.* Rs 114.

Let the rate of interest charged under instalment plan be $r$% per annum.

At the end of 5 months, Rs 114 will amount to

$$\text{Rs}\left(114 + \frac{114 \times r \times 5}{100 \times 12}\right) \qquad ...(i)$$

On the other hand at the end of 5th month :

First instalment of Rs 25 will amount to : Rs $\left(25 + \frac{25 \times r \times 4}{100 \times 12}\right)$

Second instalment of Rs 25 will amount to : Rs $\left(25 + \frac{25 \times r \times 3}{100 \times 12}\right)$

Third instalment of Rs 25 will amount to : Rs $\left(25 + \frac{25 \times r \times 2}{100 \times 12}\right)$

Fourth instalment of Rs 25 will amount to : Rs $\left(25 + \frac{25 \times r \times 1}{100 \times 12}\right)$

Fifth instalment of Rs 25 will amount to : Rs 25

At the end of fifth month, the five instalments taken together will amount to :

$$\text{Rs}\left[25 \times 5 + \frac{25 \times r}{100 \times 12}(4 + 3 + 2 + 1)\right], \textit{ i.e. } \text{Rs}\left(125 + \frac{250\,r}{1200}\right) \qquad ...(ii)$$

Thus, from (*i*) and (*ii*), we get :

$$114 + \frac{114 \times r \times 5}{100 \times 12} = 125 + \frac{250\,r}{1200}$$

Solving this equation for $r$, we get :

$$r = 41.25$$

$\therefore$ Rate = 41.25%

**Example 4 :** A television set is sold for Rs 4800 cash or for Rs 1200 cash down payment followed by five monthly equal instalments. If the rate of interest charged by the seller is 48% per annum, find each instalment.

**Solution :** Cash price of the television set = Rs 4800

In the instalment plan, the cash down payment is Rs 1200

$\therefore$ Price to be paid in instalments has its present value = Rs (4800 − 1200)
= Rs 3600

Rate = 48%

$\therefore$ At the end of 5 months, Rs 3600 will amount to

$$\text{Rs}\left(3600 + \frac{3600 \times 48 \times 5}{100 \times 12}\right) \qquad ...(i)$$

i.e. $\quad$ Rs (3600 + 720) *i.e.*, Rs 4320

Let each instalment be Rs $x$, *i.e.*

Rs $x$ be paid at the end of each month.

At the end of 5th month :

1st instalment of Rs $x$ will amount to : Rs $\left[x + \frac{x \times 48 \times 4}{100 \times 12}\right]$

2nd instalment of Rs $x$ will amount to : Rs $\left[x + \frac{x \times 48 \times 3}{100 \times 12}\right]$

3rd instalment of Rs $x$ will amount to : Rs $\left[x + \frac{x \times 48 \times 2}{100 \times 12}\right]$

4th instalment of Rs $x$ will amount to : Rs $\left[x + \frac{x \times 48 \times 1}{100 \times 12}\right]$

5th instalment of Rs $x$ will amount to : Rs $x$

At the end of 5th month, five instalments taken together will amount to

$$\text{Rs}\left[\left(x + \frac{x \times 48 \times 4}{100 \times 12}\right) + \left(x + \frac{x \times 48 \times 3}{100 \times 12}\right) + \left(x + \frac{x \times 48 \times 2}{100 \times 12}\right) + \left(x + \frac{x \times 48 \times 1}{100 \times 12}\right) + x\right]$$

i.e. $$\text{Rs}\left[5x + \frac{48x}{100 \times 12}(4 + 3 + 2 + 1)\right] \qquad ...(ii)$$

i.e. $$\text{Rs}\ \frac{27x}{5}$$

Thus, from (*i*) and (*ii*), we get

$$\frac{27x}{5} = 4320$$

i.e. $$x = 4320 \times \frac{5}{27} = 800$$

$\therefore$ Each instalment = Rs 800

## Exercises 6.1

1. An electric iron is sold for Rs 110 cash or for Rs 50 cash down payment followed by Rs 62 after a month. Find the rate of interest charged under instalment plan.
2. A bicycle is sold for Rs 400 cash or for Rs 160 cash down payment followed by 2 monthly instalments of Rs 130 each. Find the rate of interest.
3. A pressure cooker is available on Rs 180 cash or for Rs 70 cash down payment followed by Rs 60 a month for 2 months. Find the rate of interest charged under instalment plan.
4. A television set is priced at Rs 2400 cash or Rs 1200 cash down payment followed by 6 monthly instalments of Rs 225 each. What rate of interest will the dealer charge under instalment plan ?
5. A mixi is marked at Rs 1000 cash or Rs 250 cash down payment followed by Rs 200 a month for 4 months. Find the rate of interest for this instalment plan.
6. A room cooler is marked at Rs 2000 cash or Rs 400 cash down payment followed by Rs 300 per month for 6 months. Determine the rate of interest charged under this instalment plan.
7. A watch is sold either for Rs 180 cash or for Rs 40 cash down payment followed by Rs 30 a month for 5 months. Determine the rate of interest.
8. Determine the interest rate charged under each of the following instalment plans :

| Article | Cash price | Cash down payment | Each instalment | No. of monthly instalments |
|---|---|---|---|---|
| (*i*) T. V. | 2575 | 1000 | 300 | 6 |
| (*ii*) Refrigerator | 3580 | 1500 | 440 | 5 |
| (*iii*) Type-writer | 3600 | 1200 | 280 | 10 |
| (*iv*) Tape-recorder | 1600 | 300 | 175 | 8 |

9. An article is sold for Rs 100 cash or for Rs 10 as cash down payment followed by 5 equal monthly instalments. If the rate of interest charged under instalment plan be 48% per annum, determine the monthly instalment.
10. A pocket transistor is sold for Rs 125 cash or for Rs 26 as cash down payment followed by 4 equal monthly instalments. If the rate of interest charged is 25% per annum, determine the monthly instalment.
11. A ceiling fan is marked at Rs 485 cash or Rs 105 cash down payment followed by 3 equal monthly instalments. If the rate of interest charged under this instalment plan is 16% per annum, find the monthly instalment.

### 6.2 Repayment of Loans in Instalments

In the previous section, we have dealt with problems on instalment buying when the

period over which the instalment would range was in months decidedly less than a year; accordingly computations of simple interest were made. In the following problems of money lending and payment by instalments, the range normally is in years and so compound interest computations are used in each case.

In order to start a small scale industry, banks give loans to the persons. That amount is to be paid back to the bank in instalments along with the interest. Thus, for example, if a bank gives a loan of Rs 10000 which is to be paid back in 10 instalments and the bank charges an interest of 18% per annum, then the different instalments can be calculated. If the instalments are given one can find the rate of interest which the bank charges.

The following examples illustrate the problems discussed above.

**Example 1 :** A loan of Rs 12000 is obtained by Krishna from Bank of Baroda to repair her house. The amount is to be paid back in 4 annual instalments. How much is each instalment, if the rate of interest is compounded annually on the balance at 8% and is to be included in each instalment.

**Solution :** The loan is to be paid in 4 annual instalments.

$\therefore$ Krishna will pay Rs $(12000 \div 4)$, i.e. Rs 3000 every year together with the interest on the balance for 1 year.

Now, interest on Rs 12000 for 1 year

$$= \text{Rs } \frac{12000 \times 8 \times 1}{100} = \text{Rs } 960$$

$\therefore$ Instalment at the end of first year

$$= \text{Rs } (3000 + 960) = \text{Rs } 3960$$

Balance at the end of first year

$$= \text{Rs } (12000 - 3000) = \text{Rs } 9000$$

Interest on Rs 9000 for the second year

$$= \text{Rs } \frac{9000 \times 8 \times 1}{100} = \text{Rs } 720$$

$\therefore$ Instalment at the end of second year

$$= \text{Rs } (3000 + 720) = \text{Rs } 3720$$

Balance at the end of second year

$$= \text{Rs } (9000 - 3000) = \text{Rs } 6000$$

Interest on Rs 6000 for the third year

$$= \text{Rs } \frac{6000 \times 8 \times 1}{100} = \text{Rs } 480$$

$\therefore$ Instalment at the end of third year

$$= \text{Rs } (3000 + 480) = \text{Rs } 3480$$

Balance at the end of third year

$$= \text{Rs } (6000 - 3000) = \text{Rs } 3000$$

Interest on Rs 3000 for fourth year

$$= \text{Rs } \frac{3000 \times 8 \times 1}{100} = \text{Rs } 240$$

$\therefore$ Instalment at the end of fourth year

$= \text{Rs } (3000 + 240) = \text{Rs } 3240$

Hence, the four instalments are :

Rs 3960, Rs 3720, Rs 3480, Rs 3240

**Example 2**: A sum of Rs 4550 is borrowed from a money lender at 20% per annum, compounded annually. If the amount is to be paid back in three equal instalments, determine the annual instalment.

**Solution** : Let each instalment be Rs $x$.

We know that $A = P\left(1 + \frac{r}{100}\right)^n$

where $A$ = amount, $P$ = principal

$r$ = rate, $n$ = time period

$\therefore \quad P = A \div \left(1 + \frac{r}{100}\right)^n$

$\therefore$ Principal for the amount $x$, at the end of first year will be

$$x \div \left(1 + \frac{20}{100}\right)^1, \text{ i.e. } \left(\frac{5}{6}\right)^1 . x$$

Similarly, principals for the amount $x$ at the end of second year and third year will be

$\left(\frac{5}{6}\right)^2 . x$ and $\left(\frac{5}{6}\right)^3 . x$ respectively.

The sum borrowed is Rs 4550. Hence, we have

$$\left(\frac{5}{6}\right)^1 . x + \left(\frac{5}{6}\right)^2 . x + \left(\frac{5}{6}\right)^3 . x = 4550$$

Solving this equation we get :

$$x = 2160$$

$\therefore$ Each instalment = Rs 2160

**Example 3** : A man borrows money on compound interest and returns it in two equal annual instalments. If the rate of interest is 15% per annum and the yearly interest is Rs 1058 find the principal and interest charged with each instalment.

**Solution : First Method :**

Let principal be Rs $x$.

Rate = 15%

At the end of 2 years Rs $x$ amounts to

$$x\left(1 + \frac{15}{100}\right)^2, \text{ i.e. } x\left(\frac{23}{20}\right)^2 \qquad \ldots(i)$$

Now the yearly instalment is Rs 1058.

**At the end of 2nd year**

1st instalment will amount to : Rs $1058 \times \left(1 + \frac{15}{100}\right)^1$

i.e. Rs $1058 \times \left(\frac{23}{20}\right)^1$

2nd instalment will amount to : Rs 1058

At the end of 2nd year, the two instalments taken together will amount to

$$\text{Rs}\left[1058\left(\frac{23}{20}\right)^1 + 1058\right], \text{ i.e. Rs}\left(1058 \times \frac{43}{20}\right) \quad \ldots(ii)$$

Thus, from (*i*) and (*ii*), we get :

$$x\left(\frac{23}{20}\right)^2 = 1058 \times \frac{43}{20}$$

Solving this equation, we get :

$$x = 1720$$

$\therefore$ Principal = Rs 1720

Total interest charged = Rs (2116 − 1720) = Rs 396

Interest charged with 1st instalment = Rs $\left(\frac{1720 \times 15}{100}\right)$ = Rs 258

Interest charged with 2nd instalment = Rs (396 − 258)
= Rs 138

**Second Method :**

The man paid Rs 1058 as amount at the end of first year and another Rs 1058 as the amount at the end of second year.

$\therefore$ Principal for the first year = Rs $\left[1058 \div \left(1 + \frac{15}{100}\right)\right]$ = Rs 920

Principal for the second year = Rs $\left[1058 \div \left(1 + \frac{15}{100}\right)^2\right]$ = Rs 800

$\therefore$ Total principal = Rs 920 + Rs 800 = Rs 1720

Total amount paid = Rs 2116

$\therefore$ Total interest = Rs 2116 − Rs 1720 = Rs 396

Interest charged with first instalment = Rs $\left(1720 \times \frac{15}{100}\right)$ = Rs 258

Interest charged with second instalment = Rs (396 − 258) = Rs 138

**Example 4 :** Subhash purchased a refrigerator on the terms that he is required to pay Rs 1500 cash down payment followed by Rs 1020 at the end of first year, Rs 1003 at the end of second year and Rs 990 at the end of third year. Interest is charged at the rate of 10% per annum. Calculate the cash price and the total interest charged.

**Solution :** Let the cash price of the refrigerator be Rs $x$.

Cash down payment = Rs 1500

$\therefore$ Remaining amount = Rs $(x - 1500)$

1st instalment paid at the end of first year = Rs 1020

$\therefore$ Principal of 1st instalment = Rs $\left[1020 \div \left(1 + \frac{10}{100}\right)^1\right]$

$$= \text{Rs}\left[1020 \div \left(\frac{11}{10}\right)^1\right]$$
$$= \text{Rs}\left[1020 \times \left(\frac{10}{11}\right)^1\right]$$

Second instalment paid at the end of 2nd year = Rs 1003

$\therefore$ Principal of 2nd instalment = Rs $\left[1003 \times \left(\frac{10}{11}\right)^2\right]$

Third instalment paid at the end of third year = Rs 990

$\therefore$ Principal of 3rd instalment = Rs $\left[990 \times \left(\frac{10}{11}\right)^3\right]$

Thus, the principal of the three instalments taken together is

Rs $\left[1020 \times \left(\frac{10}{11}\right)^1 + 1003 \times \left(\frac{10}{11}\right)^2 + 990 \times \left(\frac{10}{11}\right)^3\right]$ and so :

$$x - 1500 = \left[\left(1020 \times \frac{10}{11}\right) + \left\{1003 \times \left(\frac{10}{11}\right)^2\right\} + \left\{990 \times \left(\frac{10}{11}\right)^3\right\}\right]$$
$$= 1020 \times \frac{10}{11} + \left(\frac{10}{11}\right)^2\left(1003 + 990 \times \frac{10}{11}\right)$$

Solving this equation, we get :

$$x = 4000$$

$\therefore$ Cash price of the refrigerator = Rs 4000

Price of the refrigerator under instalment plan

= Rs (1020 + 1003 + 990 + 1500) = Rs 4513

$\therefore$ Total interest charged = Rs (4513 − 4000)

= Rs 513

**Example 5** : One can purchase a flat from a house building society for Rs 55000 cash or on the terms that he should pay Rs 4275 as cash down payment and the rest in three equal half yearly instalments. The society charges interest at the rate of 16% per annum compounded half yearly. If the flat is purchased under instalment plan, find the value of each instalment.

**Solution** : Cash price of the flat = Rs 55000

In the instalment plan, cash down payment = Rs 4275

$\therefore$ price to be paid in instalment has its present value equal to

(Rs 55000 − Rs 4275), i.e. Rs 50725

Let each instalment be Rs $x$

Rate = 16% per annum

$\therefore$ Principal for the amount of Rs $x$ at the end of first six months

$$= \text{Rs } x \div \left(1 + \frac{8}{100}\right)^1$$
$$= \text{Rs } x \div \left(\frac{27}{25}\right)^1$$
$$= \text{Rs}\left(\frac{25}{27}\right)^1 x$$

Similarly principal for the next instalment to be paid at the end of next six months $= \text{Rs}\left(\frac{25}{27}\right)^2 x$ and principal for the third instalment to be paid after another six months $= \text{Rs}\left(\frac{25}{27}\right)^3 x$. Thus the principal of the three instalments (in rupees) taken together is $\left(\frac{25}{27}\right)^1 x + \left(\frac{25}{27}\right)^2 x + \left(\frac{25}{27}\right)^3 x$ which should be equal to 50725.

Thus, $\left(\frac{25}{27} x\right)\left[1 + \frac{25}{27} + \left(\frac{25}{27}\right)^2\right] = 50725$

Solving this equation we get $x = 19683$

$\therefore$ Each instalment is Rs 19683.

## Exercises 6.2

1. A sum of Rs 5600 is paid back in 4 yearly instalments. How much is each instalment, if the interest is compounded annually on the balance at 8% per annum and is to be included in each instalment?
2. A sum of Rs 6000 is paid back in 3 annual instalments. How much is each instalment, if the interest is compounded annually on the balance at 10% per annum and is to be included in each instalment?
3. A sum of Rs 8400 is to be returned in three annual instalments. What is the annual instalment, if the rate of interest is $9\frac{1}{2}$% per annum compounded annually on the balance and is to be included in each instalment?
4. The price of a tape-recorder is Rs 1561. A customer purchased it by paying a cash of Rs 300 and balance with due interest in 3 half yearly equal instalments. If the dealer charges interest at the rate of 10% per annum compounded half yearly, find the value of each instalment.
5. A loan of Rs 2550 is to be paid back in two equal half yearly instalments. How much is each instalment, if the interest is compounded half yearly at 8% per annum?
6. A sum of Rs 2600 is to be paid back in 2 equal annual instalments. What is the annual instalment, if the rate of interest is 8% per annum compounded annually?
7. A man borrows Rs 816 and agrees to return it in two equal annual instalments. What is the annual instalment, if the rate of interest is 12.5% per annum compounded annually?
8. Govind borrowed money from a money lender and agreed to pay back in 3 equal annual instalments of Rs 665.50 each. What sum did he borrow, if

the rate of interest charged by the money lender was 10% per annum compounded annually ?

9. A man takes loan on compound interest and returns it in two equal annual instalments. If the rate of interest is 16% per annum and the yearly instalment is Rs 1682, find the principal and the interest charged with each instalment.
10. A man borrowed some money and paid back in 3 equal annual instalments of Rs 2160 each. What sum did he borrow, if the rate of interest charged by the money lender was 20% per annum compounded annually ? Find also the total interest charged.
11. Kusum borrowed money and returned it in 3 equal quarterly instalments of Rs 4630.50 each.. What sum did he borrow if the rate of interest was 20% per annum compounded quarterly ? Find also the total interest charged.
12. Naresh took loan from a bank and the payment was made in 3 annual instalments of Rs 2600, Rs 2490 and Rs 2200 payable at the end of first, second and third year respectively. Interest was charged at 10% per annum. Calculate the amount of loan taken and the interest paid by him.
13. A person borrows Rs 5407.50 and agrees to pay the loan back with compound interest at the rate of $13\frac{1}{3}\%$ per annum in 3 equal half yearly instalments. Find the amount of each instalment, if the interest is compounded half yearly.
14. Sanjay bought a gas stove on instalment basis. He has to pay Rs 500 cash down payment and Rs 810 at the end of first year, Rs 520 at the end of second year and Rs 460 at the end of third year. Interest is charged at the rate of 15% per annum. Calculate the total cash price of the gas stove.
15. A dealer advertises that a casette recorder is sold at Rs 450 cash down followed by two yearly instalments of Rs 680 and Rs 590 at the end of first year and second year respectively. If the interest charged is 18% per annum compounded annually, find the cash price of the casette recorder.
16. A colour T.V. set is purchased under instalment purchase system. Cash down payment is Rs 2000 and 3 annual instalments of Rs 1800, Rs 1560 and Rs 1430 are payable at the end of first year, second year and third year respectively. If the rate of interest is 10% per annum compound interest, find the cash price of the T.V. set and the total interest charged under instalment plan.
17. A sewing machine is available at Rs 240 cash down payment followed by 3 annual instalments of Rs 380, Rs 240 and Rs 200 payable at the end of first year, second year and third year respectively. If the rate of interest is 25% per annum compound interest, find the cash price and total interest being charged under the instalment plan.

CHAPTER 7

# Banking

## 7.1 Introduction

In the early stages of development of civilisation, probably, man hunted or produced as much as he needed for a day or two. Later, with the development of agriculture, he started producing for a season or for a year and the practice of storing for future started. With the shift from barter to money economy, people started saving money for their later days. This resulted in the need for keeping the savings in a secured place. The rich men of the yore (of olden days), who had some credibility and also the means and might to defend themselves and their property, used to keep the savings of the common man and charged some money for this service. They also used to do money lending.

Gradually, they found that a part of money that was deposited with them for security could also be used for lending to others because not all depositors would come for withdrawal of their money simultaneously. This resulted in utilisation of the deposited money for the purpose of lending. These money lenders later started paying some money to the depositors for their deposits and charging interest at a higher rate from the borrowers ; thus, making profits utilising the deposits of the people. This ultimately led to the coming up of the institution of banking. With the passage of time this institution of banking refined itself and is in its present form as we see it today.

The bank is an institution where those who have some savings, small or big, keep their money in the form of deposits and those who need money can borrow on payment of interest with certain conditions that assure recovery of the borrowed money. The rate of interest charged by the bank from its borrowers will normally be higher than what it pays to the depositors. Banks also perform the functions of a clearing house which can be understood from the following example :

Ramesh has to pay Rs 10,000 to Suresh for the colour television set which he purchased from Suresh. Instead of paying cash, Ramesh gives Suresh a cheque for Rs 10,000 drawn on the bank in which he has his account. Suresh presents this cheque in his bank where he has his account. Suresh's bank arranges to get the money transferred from Ramesh's account to Suresh's account. Thus, a bank not only does safe keeping for depositors but also helps in many kinds of money transactions.

Some banks also issue traveller's cheques which can be cashed at different stations so that the risk of carrying money by the person is avoided. But these traveller's cheques do not earn any interest because there is no deposit with the bank.

## 7.2 Savings Bank Account

There are different types of deposits with a bank. The first and the most popular is the **Savings Bank Account**. This account can generally be opened in a bank with even a small amount of five or ten rupees only. After opening the account, the account holder can go on depositing the money into his account. He can also withdraw the money from his account whenever he needs either through a withdrawal form along with the passbook of the depositor or by a cheque. A cheque book is issued to the depositor subject to the condition that he will keep a minimum balance with the bank, in accordance with the rules of the bank.

A passbook is issued to every depositor by the bank in which datewise entries regarding his deposits, withdrawals and entries of interest that is paid are made by the bank. The general format of a bank passbook is given below :

| *Date* | *Particulars* | *Amount withdrawn* *Rs P* | *Amount deposited* *Rs P* | *Balance* *Rs P* | *Initials* |
|---|---|---|---|---|---|
| | | | | | |

(There may be minor variations in the passbook format of individual bank.)

At present the bank pays interest on the minimum balance in the savings bank account in a calendar month after the tenth day of that month. The present rate of interest for the savings bank account is 5% per annum (*p.a.*) compounded half yearly or annually according to the rules of the individual bank. Savings bank account can be opened with a Post Office also. The Post Office pays interest at the rate of 5.5% per annum.

In a passbook, money deposited is credited to the account of the depositor and money withdrawn is debited to his account.

In the following, we consider some examples to explain the ideas discussed above.

**Example 1 :** Sharda opens a savings bank account with the State Bank of India on 2.4.85 with a deposit of Rs 200. She deposits Rs 50 on 9.4.85. As the qualifying amount is the minimum balance after the tenth day of the month, she is entitled to an interest on Rs 250 for the month of April, 85 as she has neither deposited nor withdrawn any money after the 10th of April.

**Example 2 :** Joginder Pal has a savings bank account with the Central Bank of India. The entries, for the month of February, in the passbook are as follows :

| *Date* | *Particulars* | *Amount withdrawn* *Rs P* | *Amount deposited* *Rs P* | *Balance* *Rs P* |
|---|---|---|---|---|
| 1.2.85 | By cash | | 100.00 | 1100.00 |
| 7.2.85 | By cheque | | 100.00 | 1200.00 |
| 24.2.85 | By cheque | | 150.00 | 1350.00 |

What is the sum for which he will earn interest for the month of February, 1985 ?

**Solution :** Here, even though the balance on 24th February is Rs 1350, the minimum balance after 10th February is Rs 1200 and so, Rs 1200 is the principal on which Joginder Pal will earn interest for this month.

**Example 3 :** Joseph deposited Rs 350 cash on 4th July, 1984 in his savings bank account in the Syndicate Bank. On the last day of June 84, he had a balance of Rs 3000 in his account. On 9th July, 1984 he withdrew Rs 900 by cheque and again deposited Rs 50 on 10th July, 1984. He withdrew Rs 200 on 30th July, 1984 and then deposited Rs 200 on 31st July, 1984. Find the principal for which he earned interest for the month of July, 1984.

**Solution :** Let us first write the entries in the passbook and then calculate the principal for the month.

| *Date* | *Particulars* | *Amount withdrawn* *Rs P* | *Amount deposited* *Rs P* | *Balance* *Rs P* |
|---|---|---|---|---|
| 1.7.84 | | | | 3000.00 |
| 4.7.84 | By cash | | 350.00 | 3350.00 |
| 9.7.84 | To self | 900.00 | | 2450.00 |
| 10.7.84 | By cash | | 50.00 | 2500.00 |
| 30.7.84 | To self | 200.00 | | 2300.00 |
| 31.7.84 | By cash | | 200.00 | 2500.00 |

We see that the principal for which Joseph is entitled to earn interest for the month of July, 1984 is Rs 2300. On 10th July he had a balance of Rs 2500 and again on 31st July he had a balance of Rs 2500. But because of the withdrawal of Rs 200 on 30th July the minimum balance (after 10th) is reduced to Rs 2300.

## CALCULATION OF INTEREST ON SAVINGS BANK ACCOUNT

Let us learn how to calculate interest on savings bank account. Though the banks generally calculate the interest and enter it in the passbook every six months or every one year, the timing of this compounding may be different for different banks. In all examples and exercises the amount qualifying for interest in each month will be calculated on the basis of the minimum balance after 10th day of the calendar month.

**Example 4 :** Muthuswamy's savings bank account opened on 4th January, 1985 has the following entries in the passbook :

| *Date* | *Particulars* | *Withdrawals* *Rs P* | *Deposits* *Rs P* | *Balance* *Rs P* |
|---|---|---|---|---|
| 4.1.85 | By cash | | 500.00 | 500.00 |
| 8.1.85 | To self | 100.00 | | 400.00 |
| 10.1.85 | By cheque clearing | | 300.00 | 700.00 |
| 29.1.85 | To withdrawal slip | 50.00 | | 650.00 |
| 1.2.85 | By salary | | 1450.00 | 2100.00 |
| 5.2.85 | To cheque no. 837 | 700.00 | | 1400.00 |
| 15.2.85 | To cheque no. 838 | 500.00 | | 900.00 |
| 28.2.85 | To cheque no. 839 | 150.00 | | 750.00 |
| 1.3.85 | By salary | | 1450.00 | 2200.00 |
| 6.3.85 | To self | 750.00 | | 1450.00 |
| 18.3.85 | By cheque | | 100.00 | 1550.00 |
| 21.3.85 | To withdrawal slip | 200.00 | | 1350.00 |

If the rate of interest is 5% per annum and it is paid at the end of March and September every year, find the interest earned by him at the end of March on his S. B. account.

**Solution :** Principal for the month of January, 85 = Rs 650.00

Principal for the month of February, 85 = Rs 750.00

Principal for the month of March, 85 = Rs 1350.00

$\therefore$ Corresponding principal for one month = Rs (650 + 750 + 1350)
= Rs 2750.00

Rate = 5% *p.a.*

$$\therefore \quad \text{Interest} = \text{Rs}\left(\frac{2750 \times 5 \times 1}{100 \times 12}\right) = \text{Rs } 11.46 \text{ (approx.)}$$

**Example 5 :** The entries in the passbook of a S.B. account holder are as follows :

| *Date* | *Particulars* | *Withdrawals* Rs P | *Deposits* Rs P | *Balance* Rs P |
|---|---|---|---|---|
| 1983 | | | | 1700.00 |
| Jan. 5 | By cash | | 200.00 | 1650.00 |
| 27 | To cheque no. 342 | 50.00 | | 1800.00 |
| Feb. 7 | By cheque | | 150.00 | 1700.00 |
| March 19 | To cheque no. 343 | 100.00 | | 2050.00 |
| May 5 | By cash | | 350.00 | 2000.00 |
| 11 | To self | 50.00 | | 2100.00 |
| June 9 | By cheque | | 100.00 | 1900.00 |
| Sept. 6 | To cheque no. 344 | 200.00 | | |

If interest is calculated at the end of September each year and the rate of interest is 6% per annum, calculate the interest due for the period January, 1983 to September, 1983.

**Solution :** Principal for the month of :

| | |
|---|---|
| January, 83 | = Rs 1650 |
| February, 83 | = Rs 1800 |
| March, 83 | = Rs 1700 |
| April, 83 | = Rs 1700 |
| May, 83 | = Rs 2000 |
| June, 83 | = Rs 2100 |
| July, 83 | = Rs 2100 |
| August, 83 | = Rs 2100 |
| September, 83 | = Rs 1900 |
| | Rs 17050 |

$\therefore$ Corresponding principal for one month = Rs 17050

Rate = 6% *p.a.*

$\therefore$ Interest = Rs $\left(\dfrac{17050 \times 6 \times 1}{100 \times 12}\right)$ = Rs 85.25

*Note* : It may be noted here that even though there is no entry for the month of April, the minimum balance with the bank is Rs 1700.

**Example 6 :** Jai Singh has a savings bank account with the Central Bank of India. The passbook entries are as follows :

| *Date* | *Particulars* | *Withdrawals* *Rs P* | *Deposits* *Rs P* | *Balance* *Rs P* |
|---|---|---|---|---|
| Feb. 7 | By cash | | 5000.00 | 5000.00 |
| March 4 | To cheque no. 236 | 1000.00 | | 4000.00 |
| 9 | By cheque | | 2000.00 | 6000.00 |
| 17 | To cheque no. 237 | 2000.00 | | 4000.00 |
| April 8 | By interest | | 37.50 | 4037.50 |
| 10 | By cheque | | 1700.00 | 5737.50 |
| 10 | To commission | 2.50 | | 5735.00 |
| 19 | To self | 535.00 | | 5200.00 |
| June 6 | To cheque no. 238 | 1200.00 | | 4000.00 |
| August 10 | By cheque | | 1000.00 | 5000.00 |
| Sept. 9 | By cash | | 1900.00 | 6900.00 |

If interest is calculated at 5% *p.a.* and interest is compounded at the end of March and September, find the amount he gets if he wants to close the account on 3rd October of the same year.

**Solution :** Let us calculate the interest. Here interest up to the month of March has already been computed and added to the balance in April. Therefore, the interest is to be calculated with effect from April.

Principal for the month of :

April = Rs 5200.00
May = Rs 5200.00
June = Rs 4000.00
July = Rs 4000.00
August = Rs 5000.00
September = Rs 6900.00

Rs 30300.00

$\therefore$ Corresponding principal for one month = Rs 30300

Rate = 5% *p.a.*

$$\therefore \quad \text{Interest} = \text{Rs}\left(\frac{30300 \times 5 \times 1}{100 \times 12}\right) = \text{Rs } 126.25$$

As Jai Singh closes the account on 3rd October he gets Rs 6900 plus the interest of Rs 126.25, i.e. Rs 7026.25.

*Note* : It is not necessary that an account be closed only after the normal due date of calculation of the interest. For example, if in example 3 Jai Singh had wanted to close the account on 22nd September instead of 3rd October he could do so. In that case he would get the interest for the period April to August.

The corresponding principal for one month (for the period April to August) would be Rs 23400.

$$\text{Rate} = 5\% \; p.a.$$

$$\therefore \quad \text{Interest} = \text{Rs}\left(\frac{23400 \times 5 \times 1}{100 \times 12}\right) = \text{Rs } 97.50$$

Thus, he would receive on closure of the account a sum of

Rs (6900 + 97.50) i.e. Rs 6997.50.

## Exercises 7.1

1. Karim opens a S.B. account with Rs 100 on 1st January, 1985. Later he deposits Rs 50 on 8th January, 1985. He withdraws Rs 10 on 11th January, 1985. Find the sum for which he will earn interest for the month of January, 1985.
2. Sudhendu's S.B. account's passbook has the following entries:

| *Date* | | *Particulars* | *Withdrawals* *Rs P* | *Deposits* *Rs P* | *Balance* *Rs P* |
|---|---|---|---|---|---|
| April | 1 | By salary | | 1960.00 | 2300.00 |
| | 5 | To self | 250.00 | | 2050.00 |
| | 16 | To cheque no. 103 | 500.00 | | 1550.00 |
| | 28 | By cash | | 50.00 | 1600.00 |
| May | 1 | By salary | | 1960.00 | 3560.00 |
| | 2 | To Vinod | 1500.00 | | 2060.00 |
| | 25 | To Rahim | 300.00 | | 1760.00 |

Calculate the sums for which he will earn interest during the months of April and May separately.

3. Krishnamurthy's Syndicate Bank S.B. account's passbook has the following entries:

| *Date* | | *Particulars* | *Withdrawals* *Rs P* | *Deposits* *Rs P* | *Balance* *Rs P* |
|---|---|---|---|---|---|
| June | 2 | By cash | | 200.00 | 200.00 |
| | 9 | By cash | | 300.00 | 500.00 |
| July | 1 | By cheque | | 2000.00 | 2500.00 |
| | 1 | To commission | 5.00 | | 2495.00 |
| | 4 | To cheque no. 579 | 995.00 | | 1500.00 |

Calculate the sums for which interest will be earned by him during the two months.

4. Dinesh has his S.B. account in the State Bank of India. His passbook has the following entries :

| *Date* | *Particulars* | *Withdrawals* Rs P | *Deposits* Rs P | *Balance* Rs P |
|---|---|---|---|---|
| *1983* | | | | |
| Oct. 1 | | | | 700.00 |
| *1984* | | | | |
| May 3 | By cash | | 1000.00 | 1700.00 |
| 11 | To cheque no. 811 | 200.00 | | 1500.00 |
| July 1 | By cheque | | 1500.00 | 3000.00 |
| 2 | By cash | | 500.00 | 3500.00 |
| August 4 | To Sunita | 700.00 | | 2800.00 |

Rate of interest is 5% *p.a.* compounded annually at the end of September. Calculate the interest that will be due to him on 1st October, 1984 and also the balance on that date.

5. Anil has a S.B. account in the United Commercial Bank. His passbook has the following entries :

| *Date* | *Particulars* | *Withdrawals* Rs P | *Deposits* Rs P | *Balance* Rs P |
|---|---|---|---|---|
| *1983* | | | | |
| Dec. 1 | By cash | | 7000.00 | 7000.00 |
| 12 | To cheque no. 527 | 1500.00 | | 5500.00 |
| 13 | By cash | | 500.00 | 6000.00 |
| *1984* | | | | |
| Jan. 1 | By salary | | 4000.00 | 10000.00 |
| 1 | To Sunil | 2000.00 | | 8000.00 |
| 9 | By cheque | | 400.00 | 8400.00 |
| Feb. 1 | By salary | | 3000.00 | 11400.00 |
| 3 | To self | 2500.00 | | 8900.00 |
| March 17 | By salary | | 1100.00 | 10000.00 |

On 31st March, 1984 he receives his transfer order and wants to close the account. Calculate the money he receives on closing the account (interest calculated at the rate of 5% *p.a.*).

6. Salim has joined a factory which pays wages by cheque only. He opens a S.B. account on February 1 and his passbook has the following entries upto 1st April of that year :

| Date | Particulars | Withdrawals Rs P | Deposits Rs P | Balance Rs P |
|---|---|---|---|---|
| Feb. 1 | By cash | | 50.00 | 50.00 |
| 2 | By salary | | 1000.00 | 1050.00 |
| 4 | To withdrawal slip | 200.00 | | 850.00 |
| 15 | By overtime allowance | | 300.00 | 1150.00 |
| 24 | To Aslam | 100.00 | | 1050.00 |
| March 1 | By salary | | 1000.00 | 2050.00 |
| 7 | To cheque no. 211 | 500.00 | | 1550.00 |
| 21 | To cheque no. 212 | 700.00 | | 850.00 |
| 27 | To self | 400.00 | | 450.00 |
| April 1 | By salary | | 1000.00 | 1450.00 |
| 11 | By interest | | ——— | ——— |

He closes the account on 11th April. Complete the entries for 11th April at the rate of 5% *p.a.*

7. Shankar Rana has a S.B. account in State Bank of India where interest is compounded at the end of September every year. His passbook entries are as follows :

| Date | Particulars | Withdrawals Rs P | Deposits Rs P | Balance Rs P |
|---|---|---|---|---|
| *1983* | | | | |
| Oct. 4 | By cash | | 100.00 | 100.00 |
| 11 | By cheque | | 300.00 | 400.00 |
| Nov. 3 | By cash | | 200.00 | 600.00 |
| 7 | To withdrawal slip | 100.00 | | 500.00 |
| *1984* | | | | |
| Jan. 3 | By cash | | 500.00 | 1000.00 |
| March 25 | By cash | | 200.00 | 1200.00 |
| June 7 | By cash | | 300.00 | 1500.00 |
| Aug. 29 | To Amol | 100.00 | | 1400.00 |
| Oct. 3 | By interest | | ——— | ——— |

Calculate and complete the entries for October 3, 1984 if interest is paid at 5% *p.a.*

8. Anita opens a S.B. account in State Bank of India on August 1, 1983 with Rs 100. She deposits Rs 100 on the first or second day of every month till and including February 1, 1984. In between she withdraws Rs 200 on October 17, 1983 and also on January 18, 1984.

Write the entries of the passbook and calculate the interest due at the end of September, 1984 if interest is calculated at the rate of 5% *p.a.* at the end of September every year.

9. Surinder Singh's S.B. account's passbook entries are as follows :

| *Date* | *Particulars* | *Withdrawals* *Rs P* | *Deposits* *Rs P* | *Balance* *Rs P* |
|---|---|---|---|---|
| Jan. 7 | By cash | | 500.00 | 500.00 |
| March 19 | To cheque no. 319 | 100.00 | | 400.00 |
| May 24 | By cheque | | 1500.00 | 1900.00 |
| July 29 | To withdrawal slip | 200.00 | | 1700.00 |
| Sept. 2 | By cash | | 1300.00 | 3000.00 |

If interest is paid at the rate of 5% *p.a.* at the end of September every year, calculate the total amount he will get if he closes the account in October of the same year.

10. Ajmal Khan, an employee of a bank, has a savings bank account in his bank that pays him interest at the rate of 6% *p.a.* which is compounded every June and December. His passbook entries are as follows :

| *Date* | | *Particulars* | *Withdrawals* *Rs P* | *Deposits* *Rs P* | *Balance* *Rs P* |
|---|---|---|---|---|---|
| *1981* | | | | | |
| Feb. | 3 | By cash | | 500.00 | 500.00 |
| | 7 | To cheque no. 731 | 200.00 | | 300.00 |
| | 11 | By cheque | | 700.00 | 1000.00 |
| March | 1 | By salary | | 2350.00 | 3350.00 |
| | 4 | To withdrawal slip | 1500.00 | | 1850.00 |
| | 31 | To Urmil | 150.00 | | 1700.00 |
| April | 1 | By salary | | 2350.00 | 4050.00 |
| | 2 | To Sri Ram | 1800.00 | | 2250.00 |
| May | 1 | By salary | | 2350.00 | 4600.00 |
| | 3 | To Munish | 2000.00 | | 2600.00 |

Calculate the interest due at the end of June and find the balance on July 1, if he deposits a cash of Rs 100 on July 1, which is also entered immediately.

### 7.3 Other Forms of Deposits

## FIXED DEPOSITS (TERM DEPOSITS)

Savings bank (savings fund) account discussed in earlier paragraphs is one type of deposits. The other type of deposits are called Fixed Deposits. These are deposits for a fixed period of time and the depositor can withdraw his money only after the expiry of the period. Of course, in case of need the depositor can get his fixed deposit terminated earlier or get a loan from the bank under terms laid down by the bank. On these deposits the rate of interest paid is higher than that paid on savings bank account.

There are two types of these fixed deposits. The first type is based on the length of the time of the deposit.

All deposits for a minimum period of one year are treated as one category and the rate of interest paid is higher than the other category which is that of short term deposits ranging from 15 days to a period less than one year.

At present, the rates of interest on short term deposits are :

| | |
|---|---|
| 15 days to less than 3 months | 4.5% *p.a.* |
| 3 months to less than 6 months | 6% *p.a.* |
| 6 months to less than 1 year | 8% *p.a.* |

The interest on short term deposits is generally simple.

The present interest rates on fixed deposits for one year or more are as follows :

| | |
|---|---|
| 1 year and above but less than 2 years | 8.5% *p.a.* |
| 2 years and above but less than 3 years | 9% *p.a.* |
| 3 years and above but less than 5 years | 10% *p.a.* |
| 5 years and above | 11% *p.a.* |

On this category of deposits compound interest is paid and it is compounded quarterly or half-yearly depending on the rules of the individual bank, and the type of deposit.

The second type of fixed deposits is based on the mode of payment of the interest. A depositor can claim the interest on completion of every time period of compounding of interest or he permits the interest to be reinvested along with the principal. This scheme is called the Money Multiplier Scheme.

## RECURRING DEPOSITS

Another type of deposits known as Recurring or Cumulative Time Deposits, generally range for a period of one year to ten years. In these deposits, the depositor deposits a fixed amount (in multiple of Rs 5) every month for a specified number of months at the end of which he will get the sum deposited by him with compound interest. Usually bank publish tables which show the amount of monthly instalment, the number of months (in multiples of 12) for which the depositor contributes and the amount the depositor would receive on the expiry of the period. Such tables are revised from time to time depending on the changes in the rates of interest. (One such table is given on page 102.)

## TABLE

### RECURRING DEPOSIT SCHEME (FOR ACCOUNTS OPENED ON OR AFTER 26.10.1982)

| No. of instalments (in months) | *Total amount due on the completion of a period of deposit on the basis of the following regulated monthly instalments* | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 5 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
| | Amount payable (in Rupees) | | | | | | | | | | |
| 12 | 62.65 | 125.30 | 250.60 | 375.90 | 501.20 | 626.50 | 751.80 | 877.10 | 1002.40 | 1127.70 | 1253.00 |
| 24 | 131.85 | 263.70 | 527.40 | 791.10 | 1054.80 | 1318.50 | 1582.20 | 1845.90 | 2109.60 | 2373.30 | 2637.00 |
| 36 | 210.40 | 420.80 | 841.60 | 1262.40 | 1683.20 | 2104.00 | 2524.80 | 2945.60 | 3366.40 | 3787.20 | 4208.00 |
| 48 | 295.55 | 591.10 | 1182.20 | 1773.30 | 2364.40 | 2955.50 | 3546.60 | 4137.70 | 4728.80 | 5319.90 | 5911.00 |
| 60 | 400.15 | 800.30 | 1600.60 | 2400.90 | 3201.20 | 4001.50 | 4801.80 | 5602.10 | 6402.40 | 7202.70 | 8003.00 |
| 72 | 509.70 | 1019.40 | 2038.80 | 3058.20 | 4077.60 | 5097.00 | 6116.40 | 7135.80 | 8155.20 | 9174.60 | 10194.00 |
| 84 | 631.80 | 1263.60 | 2527.20 | 3790.80 | 5054.40 | 6318.00 | 7581.60 | 8845.20 | 10108.80 | 11372.40 | 12636.00 |
| 96 | 767.85 | 1535.70 | 3071.40 | 4607.10 | 6142.80 | 7678.50 | 9214.20 | 10749.90 | 12285.60 | 13821.30 | 15357.00 |
| 108 | 919.55 | 1839.10 | 3678.20 | 5517.30 | 7356.40 | 9195.50 | 11034.60 | 12873.70 | 14712.80 | 16551.90 | 18391.00 |
| 120 | 1088.60 | 2177.20 | 4354.40 | 6531.60 | 8708.80 | 10886.00 | 13063.20 | 15240.40 | 17417.60 | 19594.80 | 21772.00 |

*Note*: (*a*) The amount payable on a monthly instalment of more than Rs 100/- can be calculated in proportion to the amount given in the table above for the corresponding period.

(*b*) According to this table, for a deposit for 12 months, a compound interest is paid @ 8% *p.a.*, for deposits from 24 to 35 months @ 9% *p.a.*, for 36 months to 59 months @ 10% *p.a.* and for 60 months and above @ 11% *p.a.* (The interest is compounded and added quarterly.)

(*c*) If an instalment is not deposited in the account in a particular month, then for every Rs 10/- a penal interest @ 10 paise is charged on the delayed instalment.

These cumulative deposit schemes are often used by a large number of people to save in small amounts which get accumulated for anticipated purposes (like higher education of a child, marriage, construction of house) some years hence.

*Note*: It may be noted that under recurring deposit schemes, the amount of monthly instalment must be in multiples of Rs 5 and the number of instalments in multiples of 12.

Now let us understand how to use the table, in the following examples :

**Example 1 :** Ajay saves Rs 20 from his monthly pocket allowance and puts this saving every month in a bank recurring deposit scheme for a period of 84 months. What amount does he get on maturity ?

**Solution :** See the table on page 102.

As the monthly instalment is Rs 20 and the total number of instalments is 84 what Ajay gets after 84 months is the amount entered in the table against the row marked 84 and under the column marked 20 i.e. Rs 2527.20.

(Thus, he actually pays Rs 20 × 84 i.e. Rs 1680 and gets on maturity Rs 2527.20. He gains in this investment a sum of Rs 847.20).

**Example 2 :** Smita needs Rs 5000 after six years. She wants to invest in recurring deposit scheme on monthly instalment basis. Calculate the minimum amount of the monthly instalment which would fetch her approximately this amount after six years.

**Solution :** See the table on page 102.

As the amount needed on maturity is Rs 5000 and the time is six years, we read the table against the row marked 72. The entry nearest to Rs 5000 is Rs 5097.00. This entry is under the column marked 50. Thus, Smita must deposit Rs 50 every month to get Rs 5000 for the purpose she has in her view.

*Note*: The instalments have to be in multiples of Rs 5. An instalment of Rs 45 would fetch her only Rs 4587.30 which is short of Rs 5000. Also she need not pay an instalment of Rs 55 because the minimum instalment required to get sufficient amount is Rs 50.

**Example 3 :** Sudesh can save Rs 60 every month and feels that she should invest in a recurring deposit scheme to have at least Rs 7500. Calculate the minimum number of instalments needed for this venture.

**Solution :** In the column marked 60 there is an entry of Rs 7581.60 against the row marked 84. Thus she should deposit Rs 60 every month for 84 months to get a sum of Rs 7500. The actual amount she gets is Rs 7581.60.

**Example 4 :** Chand Khan can save Rs 55 every month. He invests this amount every month in a recurring deposit scheme for 96 months. Find the amount he will get on the maturity of the deposit.

**Solution :** In the table, in the row against 96, there are two entries below Rs 50 and Rs 60. These are Rs 7678.50 and Rs 9214.20.

The amount for an instalment of Rs 55 will be the mean of these two amounts i.e.

$$\text{Rs } \left(\frac{7678.50 + 9214.20}{2}\right)$$
$$= \text{Rs } 8446.35.$$

Alternatively, in the row against 96, there are two entries below Rs 5 and Rs 50. These are Rs 767.85 and Rs 7678.50. The sum of these two is Rs 8446.35.

## Exercises 7.2

*Note* : In the following questions, use the recurring deposit table given on page 102.

1. Kavita saves Rs 35 every month and puts it in a recurring deposit scheme for 36 months. Find, with the help of the table, the amount she will get on maturity.
2. Nambiar saves Rs 40 every month and puts it in the recurring deposit scheme for 84 months. After two years he decides to contribute an additional Rs 50 in a recurring deposit scheme for a period of five years. Find the two amounts that he gets on maturity of the two schemes.
3. Banshidhar puts in Rs 50 every month for a period of 7 years in a recurring deposit scheme. Find the amount he gets on maturity.
4. Ram Nath starts contributing Rs 15 per month to a recurring deposit scheme. What should be the minimum number of instalments in order to get at least Rs 1200 on maturity ?
5. Harish wants to save a sum of at least Rs 10000 by saving Rs 50 every month in a recurring deposit scheme. Find the minimun number of instalments needed for the purpose.
6. Two brothers Ram and Shyam start saving in monthly recurring deposit schemes by contributing Rs 5 and Rs 10 every month with a target of getting an amount of Rs 1000 at least. Find how much more time Ram takes in comparison to Shyam to achieve the target.
7. Ramaswamy needs Rs 5000 after six years. Find the amount of the monthly instalment that will be just sufficient to achieve the target.
8. Kul Bhushan can save a sum of a little over Rs 10000 by depositing Rs 50 every month in a recurring deposit scheme for 120 months. In case his brother wants to save a little over Rs 10000 in seven years, what amount of monthly instalment must his brother deposit ?

CHAPTER 8

# Taxes

## 8.1 Introduction

Modern economy rests on the economy of growth which demands resources. An egalitarian society in a welfare state demands bridging the gap between the masses and the classes. The State (Government) has to spend a lot of money for various purposes like maintenance of law and order, defence, development, education, health, etc. The Government in turn has to collect money from the citizens, and in some cases non-citizens (like tourists and other visitors) to meet the expenditure it has to incur in different fields. This money that the government collects from the people is obtained through different types of taxes like : **income tax, wealth tax**, **gift tax, sales tax**, **excise and customs duty, property tax**, etc. All these taxes are proposed, enhanced or decreased in the Central Government's budget or State Government's budget, every year.

Central Government levies taxes like income tax, wealth tax, gift tax, central excise and customs, stamp duty, central sales tax, etc. State Governments levy state excise, entertainment tax, agricultural revenue tax, etc. There are certain taxes levied by the local bodies like Municipal Corporations, Municipal Committees, District Boards, Cantonment Boards, etc. Some of such taxes as are imposed by local bodies are property tax, octroi, professional tax, education cess, etc.

The areas in which the three i.e. Central Government, State Governments and Local Bodies can legislate and impose taxes are well defined. As the Central Government receives the maximum revenue from taxes, it allocates to each state/union territory a portion of the resources in the central pool for development or other expenditure.

Thus, we see that taxes are the contribution of the citizens towards the expenditure of the State in its multifarious economic and non-economic activities. The system of taxes indicate the approach to the management of the economy. The budget is the statement of Government's fiscal policy. We now propose to study very briefly the different types of taxes.

## 8.2 Direct Taxes

These are taxes imposed on individual or groups of individuals which affect them directly e.g. income tax, wealth tax, gift tax, etc.

**Income Tax**

This is a tax imposed on the income of a person or a group of persons. In the following paragraphs, we shall discuss the main features of income tax with reference to individual's income only.

Every individual whose annual income exceeds a specified limit is required by law to pay a part of his income to the Government. This is known as **income tax** and is imposed annually, by law, in the beginning of the financial year.

**Wealth Tax**

Wealth tax is the tax levied on the wealth of an individual. This is also required to be paid annually. Wealth includes all assets of an individual, irrespective of the place of location, held on the date of valuation which at present is 31st March of the previous year as defined in the Income Tax Act. In this case also only individuals owning wealth beyond a certain limit are taxed. The debts to be paid by the individual are given consideration for defining wealth. The exemption limit is Rs 5,00,000 at present.

**Gift Tax**

Gift has been defined under the Gift Tax Act, 1958. Gift means the transfer by one individual to another of any existing movable or immovable property made voluntarily without consideration to money or money's worth.

The Gift Tax Act levies a charge for every assessment year commencing on and from 1st April, 1958 in respect of gifts made by an individual during the previous year at the rates prescribed in the schedule to the Act.

There are different types of exemptions provided in the Act.

## 8.3 Computation of Income Tax

The following are some of the important points that should be kept in view while calculating the income tax :

1. Income tax is charged according to the rates prescribed for the assessment year as given in the Finance Act.
2. The tax is charged on the assessee's total income of the previous year calculated according to the provisions of the Act.
3. Even when the tax is deducted at source or paid in advance, liability to pay tax rests with the assessee by virtue of the Act.
4. Every individual, who is an assessee, is expected to file a statement of income in the prescribed form. This is known as **Income Tax Return**.

According to the existing provisions of the Income Tax Act, salaried persons having a salary income not exceeding Rs 24000 and with income from interests and dividends being less than Rs 7000 are not required to file their income tax return provided tax is deducted at source.

### Admissible Deductions

Each person is allowed certain amount of his income free of tax under the heading 'Admissible deductions.'

### (1) House Rent Allowance

Any employee who receives house rent allowance in compensation to the rent paid for residential accommodation is provided relief from income tax which is the minimum of the following sums :

(*a*) The actual amount of house rent allowance (HRA) received by the assessee in respect of the relevant period.

(*b*) Actual expenditure on rent in excess of 10% of the salary due to the assessee in respect of the relevant period.

(*c*) (*i*) 20% of the salary due to the assessee in respect of the relevant period, where the residential accommodation is situated at Ahmedabad, Bangalore, Bombay, Calcutta, Delhi, Hyderabad, Kanpur, Madras or Pune; and

(*ii*) 10% of the salary due to the assessee in respect of the relevant period where the residential accommodation is situated at any other place.

For this purpose

(*i*) 'Salary' includes dearness allowances, if the terms of employment so provide, but excluding all other allowances.

(*ii*) 'Relevant period' means the period during which the said accommodation was occupied by the assessee during the previous year.

### (2) Standard Deduction

Salaried taxpayers are allowed a consolidated standard deduction upto a maximum of Rs 6000, admissible in respect of travelling, books, taxes on profession and expenditure incurred in connection with the performance of duties in a year. In the case of employees who are not in receipt of conveyance allowance or not provided with a conveyance, standard deduction will be an amount equal to 25% of the salary subject to a maximum of Rs 6000.

### (3) Deductions on Account of Life Insurance Premium, Provident Fund, Some National Savings Certificates, etc. (*Under Section 80 C*)

The Income Tax Act provides a relief by way of deduction of certain sums in respect of the following sums paid in the previous year by the assessee out of his income chargeable to income tax, towards life insurance premium, provident fund, etc.

In computing the total income of an assessee, there shall be a deduction, in accordance with and subject to the provisions of this section, of an amount calculated, with reference to the aggregate of the sums specified in sub-section (2), at the following rates, namely :

| | |
|---|---|
| (*a*) Where such aggregate does not exceed Rs 6000 | The whole of such aggregate. |
| (*b*) Where such aggregate exceeds Rs 6000 but does not exceed Rs 12000 | Rs 6000 plus 50% of the amount by which such aggregate exceeds Rs 6000. |
| (*c*) Where such aggregate exceeds Rs 12000 | Rs 9000 plus 40% of the amount by which such aggregate exceeds Rs 12000. |

**(4) Deductions Under Section 80 G**

**1. Deductions in respect of Donation to Certain Funds, Charitable Institutions, etc.**

In computing the total income of an assessee, there shall be a deduction in accordance with and subject to the provision of this section :

(*i*) in a case where the aggregate of the sums specified in sub-section (2) includes any sum or sums of the nature specified in sub-clause (*iii a*) of clause (*a*) thereof, an amount equal to the whole of the sum or, as the case may be, sums of such nature plus fifty percent of the balance of such aggregate ; and

(*ii*) in any other case, an amount equal to fifty percent of the aggregate of the sums specified in sub-section (2).

2. The sums referred to in sub-section (1) shall be the following, namely :—

(*a*) any sums paid by the assessee in the previous year as donation to :

(*i*) the National Defence Fund set up by the Central Government ; or

(*ii*) the Jawahar Lal Nehru Memorial Fund ; or

(*iii*) the Prime Minister's Drought Relief Fund ; or

(*iii a*) the Prime Minister's National Relief Fund ; or

(*iii b*) the National Children's Fund ; or

(*iii c*) the Indira Gandhi Memorial Trust ; or

(*iv*) any other fund or any institution to which this section applies ; or

(*v*) the Government or any local authority, to be utilised for any charitable purpose, etc.

There are other deductions that are allowed to an assessee but these are not of common use and are beyond the scope of this text and so are not discussed here.

The latest rates of income tax as amended by the Finance Act, 1985 are given below :

Rates of tax on individual's taxable income :

| *Slab* | *Assessment year 1986-1987* (*Financial year 1985-86*) |
|---|---|
| Up to Rs 18,000 | Nil |
| Rs 18,000 to Rs 25,000 | 25% |
| Rs 25,001 to Rs 50,000 | 30% |
| Rs 50,001 to Rs 1,00,000 | 40% |
| Rs 1,00,001 onwards | 50% |

Let us calculate the income tax to be paid by an assessee. (*Note* : The rules given above are for the assessment year 1986-87. These may be revised and thus at any time

the information in the book may become obsolete. All calculations in the following problems are according to the rules applicable to assessment year 1986-87.)

**Example 1** : Gopal has gross annual income of Rs 30,000 (exclusive of House Rent Allowance). He deposits in his provident fund (P.F.) a sum of Rs 200 p.m. Calculate the amount of income tax he has to pay.

**Solution** : Total income = Rs 30000

Standard deduction (subject to a maximum of Rs 6000)

$$= \text{Rs } \left(\frac{30000 \times 25}{100}\right)$$

$$= \text{Rs } 7500 \text{ limited to Rs } 6000$$

Contributions to P.F. = Rs $200 \times 12$
= Rs 2400

Net taxable income = Rs 30000 − (Rs 6000 + Rs 2400)
= Rs 21600

Tax up to Rs 18000 = nil

Tax for Rs (21600 − 18000)

$$= \text{Rs } \left(\frac{3600 \times 25}{100}\right)$$

$$= \text{Rs } 900$$

Total tax to be paid = Rs 900

**Example 2** : Neena has a total income of Rs 40,000 per annum (exclusive of HRA). She deposits Rs 500 p.m. to her provident fund and pays an annual premium of Rs 2000 on life insurance. Calculate the amount of income tax she is required to pay.

**Solution** : Neena's total annual income = Rs 40000

Standard deduction (maximum) = Rs 6000

Contribution to provident fund (annual) = Rs 6000 ...(*i*)

Life Insurance premium = Rs 2000 ...(*ii*)

∴ Deductions because of (*i*) and (*ii*)

$$= \text{Rs } 6000 + \text{Rs } 2000 \times \frac{50}{100}$$

$$= \text{Rs } 7000$$

Net taxable income = Rs 40000 − Rs 6000 − Rs 7000
= Rs 27000

Income tax upto Rs 18000 = nil

Income tax for Rs 7000 (above Rs 18000 and upto Rs 25000)

$$= \textbf{Rs } \left(\frac{7000 \times 25}{100}\right) = \textbf{Rs 1750}$$

Income tax for Rs 2000 (above Rs 25000)

$$= \text{Rs} \left(\frac{2000 \times 30}{100}\right) = \text{Rs } 600$$

Total tax payable $= \text{Rs } 1750 + \text{Rs } 600 = \text{Rs } 2350$

**Example 3** : Rohini has an annual income of Rs 55000 (exclusive of HRA). Her savings in terms of provident fund are Rs 12000. She pays an annual premium of Rs 3000 for her life insurance and invests Rs 5000 in national savings certificates. In addition, she contributes Rs 1500 to Prime Minister's Fund that carries a rebate of 100% under section 80 G. Find the total income tax she is required to pay.

**Solution** : Rohini's total income (annual) = Rs 55000

Standard deduction = Rs 6000 ...(*i*)

(25% subject to a maximum of Rs 6000)

Total payment under provident fund, life insurance and national savings certificates

$$= \text{Rs } (12000 + 3000 + 5000) = \text{Rs } 20000$$

$\therefore$ Permissible deductions under section 80 C

$$= \text{Rs } 6000 + \text{Rs}\left(6000 \times \frac{50}{100}\right) + \text{Rs}\left(8000 \times \frac{40}{100}\right)$$

$$= \text{Rs } 12200 \qquad \ldots(ii)$$

Permissible deductions under section 80 G = Rs 1500 ...(*iii*)

Total permissible deductions $[(i) + (ii) + (iii)]$ = Rs (6000 + 12200 + 1500)

= Rs 19700

Net taxable income = Rs 55000 − Rs 19700

= Rs 35300

Income tax on first Rs 18000 = nil

Income tax on next Rs 7000

(in the slab Rs 18001 to Rs 25000) $= \text{Rs}\left(\frac{7000 \times 25}{100}\right) = \text{Rs } 1750$

Income tax on next Rs 10300 (slab Rs 25001 to Rs 50000)

$$= \text{Rs}\left(\frac{10300 \times 30}{100}\right) = \text{Rs } 3090$$

Total tax = Rs 1750 + Rs 3090 = Rs 4840

*Note* : In any problem on Income tax computation, the net taxable income is rounded off to the nearest 10 rupees and the tax is rounded off to the nearest rupee.

## Exercises 8.1

1. Patil has an annual income of Rs 28000 of which Rs 2520 is house rent allowance which is free of tax. His contribution to provident fund is at the rate of Rs 150 p.m. Calculate the income tax he is required to pay.
2. Subramaniam's income totals to Rs 34200 per annum exclusive of house rent allowance. His contribution to provident fund is Rs 3000 for the whole year. The total annual premium paid for life insurance is Rs 1600. Calculate the income tax to be paid by him.
3. Sudhendu has an average income of Rs 3200 p.m. exclusive of HRA. His contribution to provident fund, LIC, etc. is Rs 700 p.m. Compute the income tax he will be required to pay.
4. Rajinder has a total annual income of Rs 51000, exclusive of HRA. He pays a premium of Rs 3000 to the LIC and contributes Rs 8000 to the provident fund. In addition, he donates Rs 5000 to the Prime Minister's Fund carrying 100% deduction under section 80G. Compute the income tax he will be required to pay.
5. Rahmat Khan has an annual income of Rs 35000 exclusive of any house rent allowance. His contribution to the provident fund is Rs 150 p.m. and the annual premium for life insurance is Rs 6000. He gives a donation of Rs 1000 for which he is allowed a deduction of 50% of the aggregate donation. Calculate the income tax that he has to pay.
6. Ram Prasad Agarwal's annual income is Rs 150000 as salary and allowances, exluding house rent allowance. His savings in terms of provident fund and and LIC premium come to Rs 30000. He donates a sum of Rs 2000 to a charitable institution thus earning a relief at the rate of 50% of the donation. Calculate the income tax to be paid.
7. Pritam Singh has a total annual income of Rs 57000 excluding HRA. His contribution to provident fund is Rs 600 p.m. His LIC premium amounts to Rs 2000 *p.a.* He invests Rs 4000 in national savings certificates and cumulative time deposits. He donates Rs 3800 to the Gurudwara and earns a relief of 50% of the donation. Calculate the income tax due to him in the last month of the year if his earlier deductions for income tax are @ Rs 500 p.m.
8. John's annual income is Rs 47000 exclusive of HRA. He contributes Rs 500 p.m. to the provident fund and pays a premium of Rs 450 p.m. for life insurance. He invests Rs 3600 in national savings certificates. He gives a donation of Rs 600 to the Prime Minister's Relief Fund earning a relief of 100% on donation. His monthly deduction for income tax is Rs 250. Find the amount of income tax he has to pay in the last month of the year.

*Note* ; Problems involving taxable house rent allowance have been deliberately avoided as the computations involved become complex as for the comprehension level of the students.

## 8.4 Computation of Sales Tax

On the purchase of some items the purchaser has to pay certain amount at specified rate. This is called *sales tax*. It is an indirect tax.

Calculation of sales tax is very simple as it involves the use of the mathematical concept of percentage only. Different articles are taxed at different rates. We illustrate the calculations in the following examples :

**Example 1** : Nageshwar Rao purchases a pair of chappals from the Bata shop for Rs 99.95. The rate of sales tax is 7%. Calculate the total amount to be paid by the buyer.

**Solution** : Sale price of the pair of chappals = Rs 99.95

Rate of S.T. = 7%

$\therefore$ Amount of sales tax = Rs $\left(\frac{99.95}{100} \times 7\right)$

= Rs 7.00 (approx.)

$\therefore$ Total amount to be paid = Rs 99.95 + Rs 7.00

= Rs 106.95

**Example 2** : Rajan goes to a departmental store and buys the following articles :

(*i*) Biscuits and bakery products (S.T. @ 5 %) for Rs 25

(*ii*) Toys (S.T. @ 10%) for Rs 39

(*iii*) Medicines (S.T. @ 10%) for Rs 45

(*iv*) Clothes (S.T. @ 1%) for Rs 200

Calculate the total amount he has to pay to the departmental store.

**Solution** : Cost of biscuits, etc. = Rs 25

S.T. @ 5% = Rs $\left(\frac{25 \times 5}{100}\right)$ = Rs 1.25

Cost of toys = Rs 39

S.T. @ 10% = Rs 3.90

Cost of medicines = Rs 45

S.T. @ 10% = Rs 4.50

Cost of clothes = Rs 200

S.T. @ 1% = Rs 2

**Total amount to be paid** = Rs 320.65

## Exercises 8.2

1. Tilak buys a pair of shoes costing Rs 370. The rate of sales tax is 7%. Calculate the total amount he has to pay.
2. Karim buys bakery products for Rs 15, the rate of S.T. being 5%. He also purchases tinned food for Rs 25 on which the rate of S.T. is 10%. He gives a 100-rupee note to the shopkeeper. Find the balance returned to him.
3. Gyaneshwar visits a departmental store and purchases the following articles :
   (*i*) One rain coat for Rs 300 plus S.T @ 10%
   (*ii*) One pair of shoes for Rs 230 plus S.T. @ 9%
   (*iii*) Food articles for Rs 150 plus S.T. @ 5%
   (*iv*) Clothes for Rs 400 plus S.T. @ 1%.

   Calculate the total amount of the bill.
4. Ajay buys a motor cycle for Rs 9790 including S.T. If the rate of S.T. is 10%, what is the sale price of the motor cycle ?
5. Smita buys a leather coat costing Rs 990. The rate of S.T. is 10%. She tells the shopkeeper that he should reduce the price to such an extent that she has not to pay anything more than Rs 990 including S.T. Find the reduction needed in the cost price of the coat.

CHAPTER 9

# Similar Triangles

## 9.1 Introduction

We have studied in Class IX the congruence of two geometric figures. Two geometric figures in a plane are congruent if, without bending, twisting or stretching we can superimpose one figure on the other. In fact, when we compare two geometric figures in a plane with reference to their shapes and sizes, there are three possibilities and accordingly they fall in one of the categories below:

(*i*) The figures have neither the same shape nor the same size. For example, a circle and a square.

(*ii*) The figures have the same shape and size. For example, two circles of the same radius.

(*iii*) The figures have the same shape but not necessarily the same size. For example, two circles of different radii.

We may recall that figures of category (*ii*) are called **congruent figures**. We will now introduce the study of figures of category (*iii*). The figures of this category are called **similar figures**. For similar figures, it is not necessary that they must be of different sizes but it is necessary that they must be of the same shape. In other words, two figures are said to be similar if one of them is an exact scale model of the other. It is obvious that two congruent figures are similar but the converse is not necessarily true, namely that two similar figures need not necessarily be congruent.

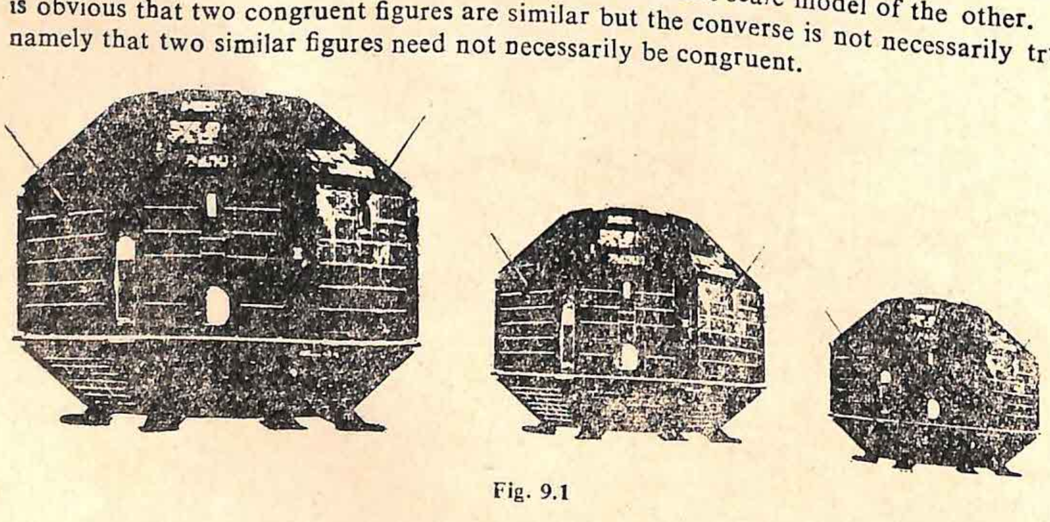


Fig. 9.1

Photographs of different sizes (Fig. 9.1) obtained from the same negative, different size prints of movies for projection on different screens, etc. are some examples of similar figures that we come across in our day-to-day life.

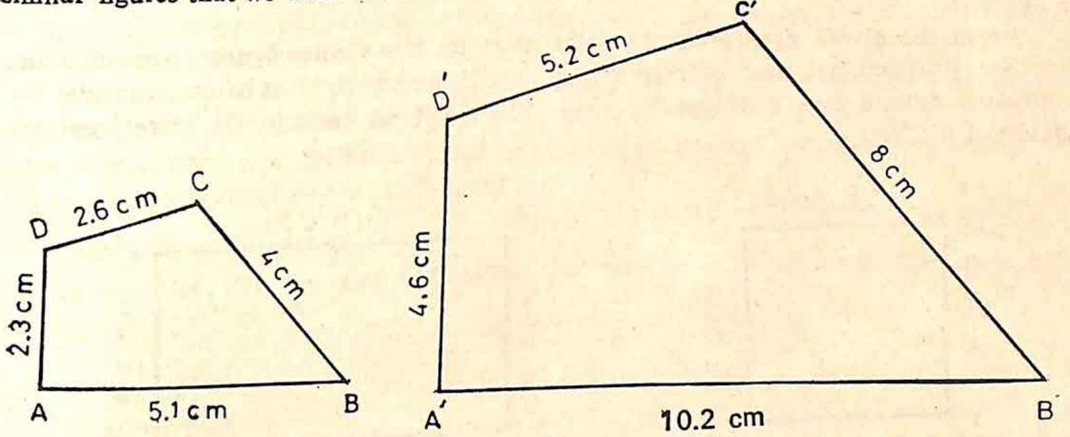



Fig. 9.2

In Fig. 9.2, $ABCD$ is a quadrilateral and $A'B'C'D'$ is the enlarged (stretched) form of the quadrilateral $ABCD$. Each side of the quadrilateral $A'B'C'D'$ is double the corresponding side of the quadrilateral $ABCD$. It means that the ratio between the corresponding sides of two quadrilaterals is the same and is 1 : 2. The corresponding angles in two quadrilaterals are equal. Similarly, in Fig. 9.3, the corresponding sides of the two triangles $ABC$ and $A'B'C'$ are proportional, the ratio being 2 : 3 and the corresponding angles are equal.

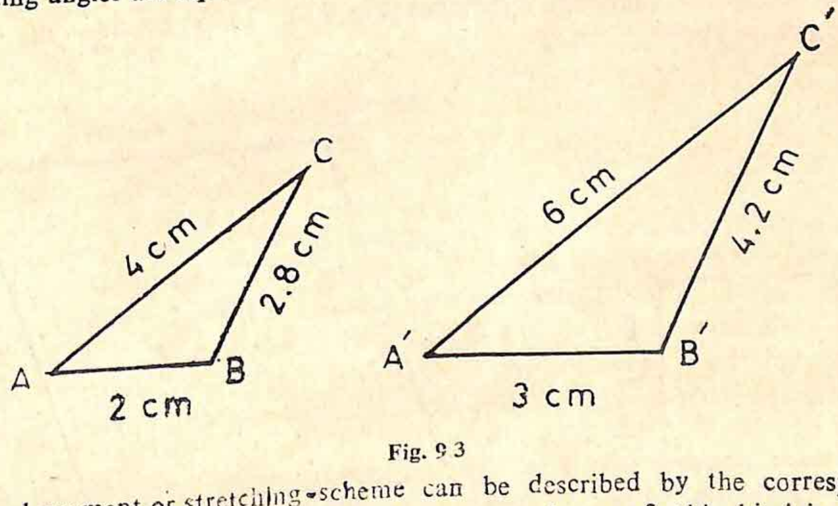



Fig. 9.3

The enlargement or stretching-scheme can be described by the correspondence $ABCD \leftrightarrow A'B'C'D'$ or $ABC \leftrightarrow A'B'C'$. A correspondence of this kind is called a **similarity**. Similar figures can be obtained by magnifying or shrinking the original figure.

For example in Fig. 9.3, $\triangle A'B'C'$ can be obtained by enlarging or magnifying $\triangle ABC$. Also $\triangle ABC$ can be obtained by diminishing or shrinking $\triangle A'B'C'$. It means that if any figure $A$ is similar to another figure $B$, then it implies that figure $B$ is also similar to the figure $A$.

From the above examples, we infer that in two similar figures, corresponding sides are proportional and corresponding angles are equal. Let us now consider for example a square and a rectangle (Fig. 9.4). Let us consider the correspondence $ABCD \leftrightarrow A'B'C'D'$.

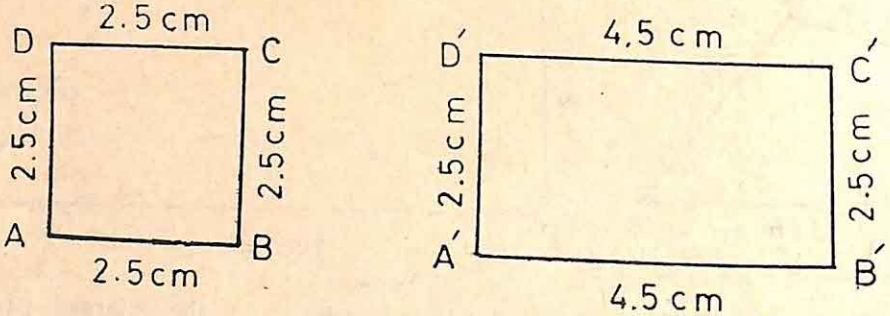



Fig. 9.4

We notice that the corresponding angles in the two figures are equal as all the angles are right angles. But corresponding sides are not proportional. Clearly neither of the two figures is a scale model of the other and hence these two figures are not similar. The fact that the corresponding sides are not proportional, is the reason for the two figures not having the same shape.

Now, consider a square and a rhombus (Fig. 9.5). Let us consider the correspondence $PQRS \leftrightarrow P'Q'R'S'$.

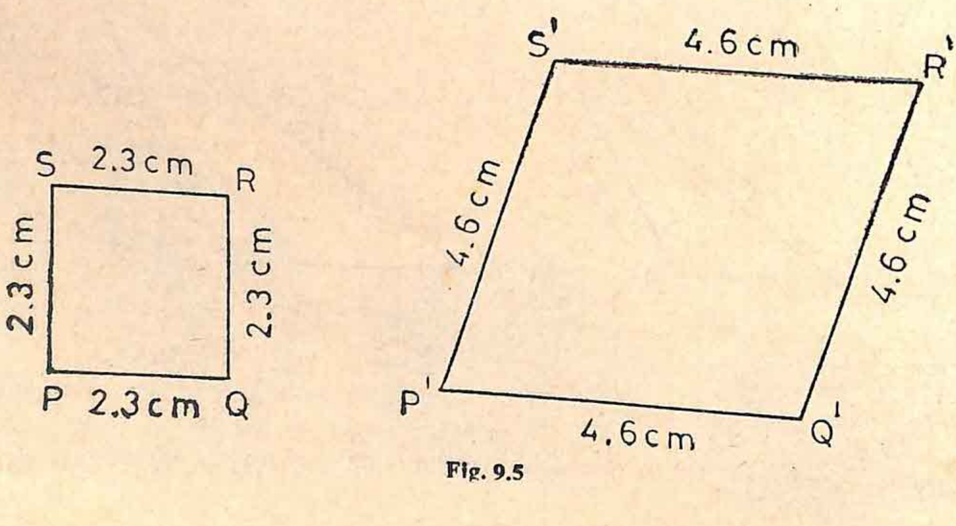



Fig. 9.5

Here, the corresponding sides of two figures are proportional (ratio being 1 : 2) but the angles are not equal as can easily be seen. You can see that the two figures do not have the same shape and so they are not similar figures. In this case, the fact that the corresponding angles of the two figures are not equal is the reason for the two figures not having the same shape.

The discussion leads us to the conclusion that for two figures to be similar (*i*) corresponding angles must be equal, and (*ii*) corresponding sides must be proportional. Later in this chapter, we will notice that in case the figures are triangles, either of the two conditions will suffice for their being similar.

## 9.2 Similarity of Triangles

Let us consider two triangles $ABC$ and $A'B'C'$ (see Fig. 9.6) with the correspondence $ABC \leftrightarrow A'B'C'$.

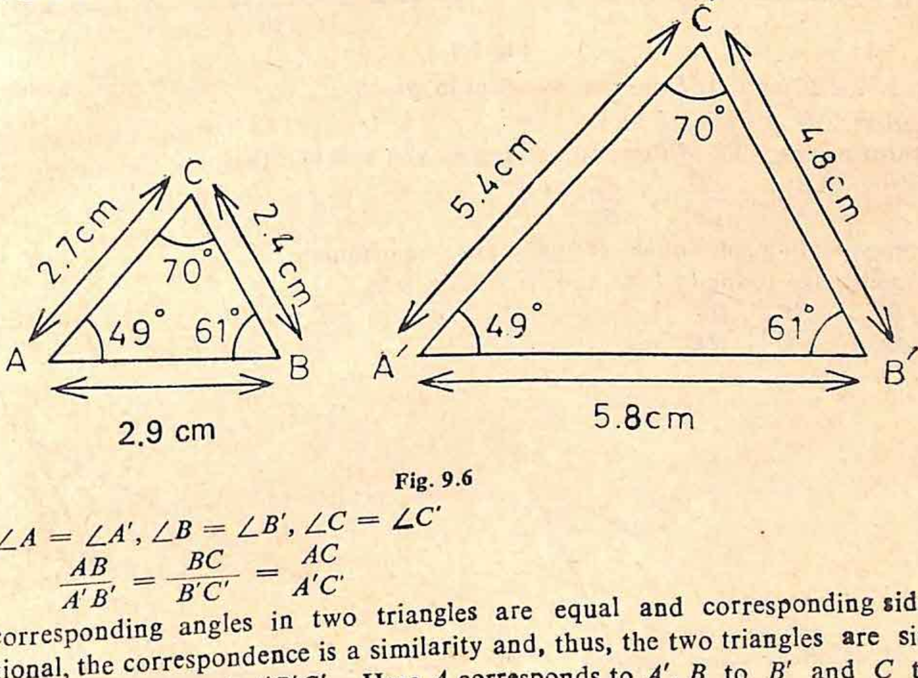



Fig. 9.6

Here $\angle A = \angle A', \angle B = \angle B', \angle C = \angle C'$

and $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$

Since corresponding angles in two triangles are equal and corresponding sides are proportional, the correspondence is a similarity and, thus, the two triangles are similar. We write it as $\triangle ABC \sim \triangle A'B'C'$. Here $A$ corresponds to $A'$, $B$ to $B'$ and $C$ to $C'$. It will be incorrect to write it as $\triangle ABC \sim \triangle B'A'C'$ or $\triangle B'C'A'$ or $\triangle A'C'B'$, etc. (Why ?). We, of course, can write it as $\triangle BAC \sim \triangle B'A'C'$ i.e. order of vertices can be arranged in such a manner that the same pair of vertices correspond (as in congruence).

If two triangles have corresponding angles equal and corresponding sides proportional, then by definition, the triangles are similar. As said earlier, in the case of triangles only one condition—either the corresponding angles being equal or the corresponding sides being proportional is sufficient for establishing the similarity of two

triangles. This can be verified by constructing two triangles with corresponding angles equal (see Fig. 9.7) and two triangles with corresponding sides proportional (see Fig. 9.8).

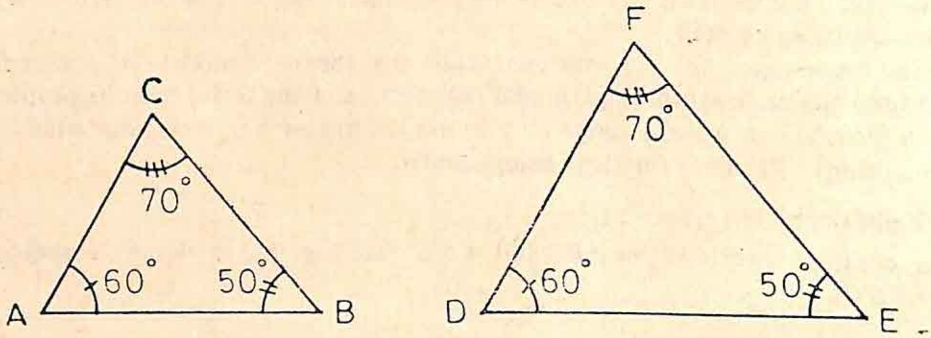



Fig. 9.7

In Fig. 9.7, $ABC$ and $DEF$ are two triangles in which

$\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F.$

On measuring the sides of these two triangles, you will find that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF},$$

i.e. corresponding sides of the triangles are proportional.

Again, take triangles $DEF$ and $HGK$ (Fig. 9.8),

where $\dfrac{DE}{HG} = \dfrac{EF}{GK} = \dfrac{DF}{HK}$.

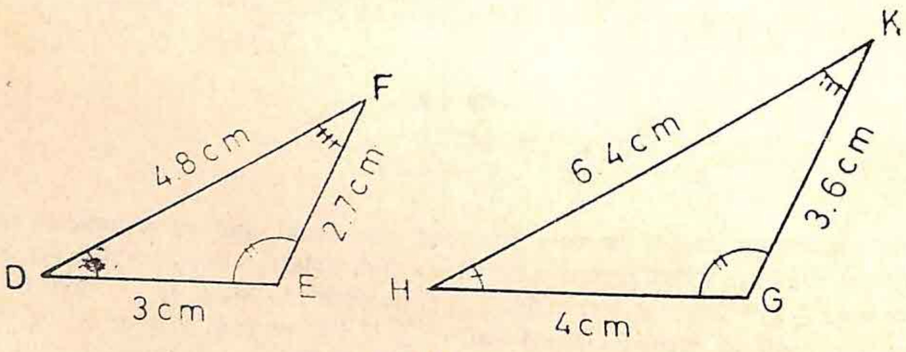



Fig. 9.8

Now measure the angles of the two triangles. You will find that $\angle D = \angle H, \angle E = \angle G$ and $\angle F = \angle K$, i.e. the corresponding angles of the triangles are equal.

Thus, we have

(*i*) **Two triangles are similar if the corresponding angles are equal.**

This is known as AAA (angle-angle-angle) criterion of similarity of triangles.

(*ii*) **Two triangles are similar if the corresponding sides are proportional.**

This is known as SSS (side-side-side) criterion of similarity of triangles.

*Note* : We know that the sum of the three angles of a triangle is 180°. Thus, to determine whether two triangles are similar or not, it is enough to verify whether two angles of one triangle are equal to the two angles of the other triangle or not. This reduces AAA criterion to just **AA (angle-angle) criterion of similarity.**

There is another criterion to determine the similarity of triangles called the **SAS (side-angle-side) criterion** which is :

(*iii*) **If one angle of a triangle is equal to one angle of the other and the sides including these equal angles are proportional, then the triangles are similar.**

This can be proved after proving the following result on **proportionality** :

**Theorem (9.1) : If a line is drawn parallel to one side of a triangle intersecting the other two sides, then the other two sides are divided in the same ratio.**

*Given* : $\triangle ABC$ and line $l$ parallel to $BC$ intersecting $AB$ at $X$ and $AC$ at $Y$ (Fig. 9.9).

*To prove* : $\frac{AX}{XB} = \frac{AY}{YC}$

*Construction* : Join $X$ to $C$ and $Y$ to $B$. Draw $YZ$ perpendicular to $AB$.

*Proof* : Area $(\triangle AXY) = \frac{1}{2} AX \cdot YZ$

Area $(\triangle XBY) = \frac{1}{2} XB \cdot YZ$

Therefore,

$$\frac{\text{Area } (\triangle AXY)}{\text{Area } (\triangle XBY)} = \frac{AX}{XB} \qquad ...(i)$$

Similarly,

$$\frac{\text{Area } (\triangle AXY)}{\text{Area } (\triangle XCY)} = \frac{AY}{YC} \qquad ...(ii)$$

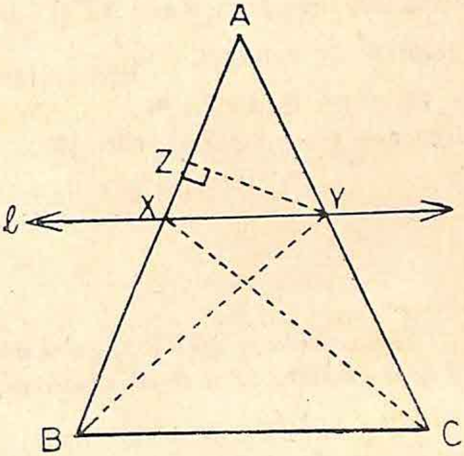



Fig. 9.9

Since $\triangle s$ $XBY$ and $XCY$ are on the same base and between the same parallels,

area $(\triangle XBY)$ = area $(\triangle XCY)$...(*iii*)

From (*i*), (*ii*) and (*iii*), it follows that

$$\frac{AX}{XB} = \frac{AY}{YC} \qquad ...(iv)$$

which is what is required to be proved.

This theorem is referred to as **Basic Proportionality Theorem.**

From (*iv*) above, we also get

$$\frac{XB}{AX} = \frac{YC}{AY}$$

Adding 1 to both sides, we obtain

$$1 + \frac{XB}{AX} = 1 + \frac{YC}{AY}$$

*i.e.* $$\frac{AX + XB}{AX} = \frac{AY + YC}{AY}$$

*i.e.* $$\frac{AB}{AX} = \frac{AC}{AY}$$

As a direct consequence of this theorem, it follows that:

**The line drawn parallel to the side $BC$ of $\triangle ABC$ through the mid-point $X$ of $AB$, bisects the side $AC$.**

The converse of the theorem is also true. It is stated below (without proof):

**Theorem (9.2) : If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side.**

In other words, if in $\triangle ABC$, $l$ is a line intersecting the sides $AB$ and $AC$ at $X$ and $Y$ respectively such that $\frac{AX}{XB} = \frac{AY}{YC}$ (see Fig. 9.10), then $XY \parallel BC$.

Let us now prove the $SAS$ criterion.

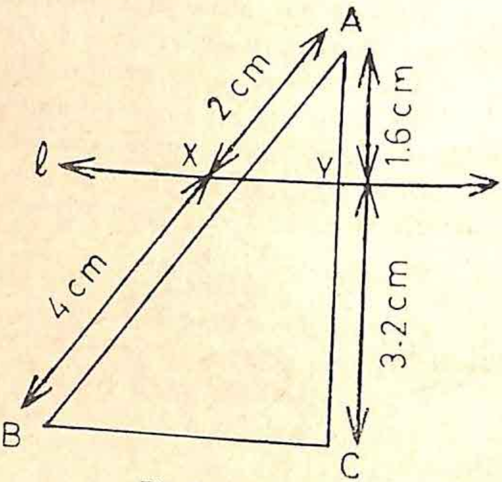



Fig. 9.10

**Theorem (9.3) : If one angle of a triangle is equal to one angle of the other and the sides including these equal angles are proportional, then the triangles are similar.**

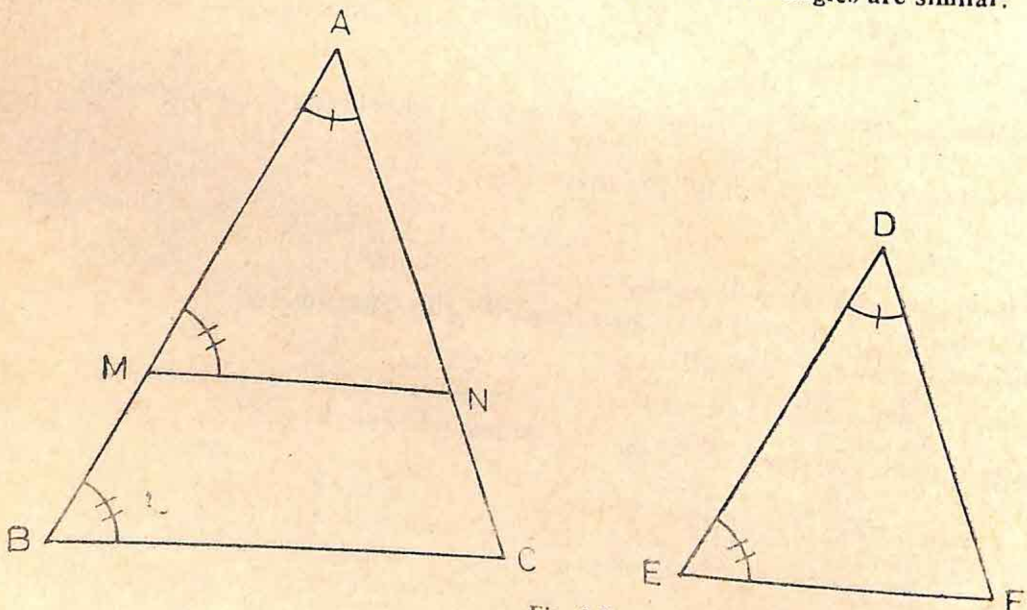



Fig. 9.11

*Given* : In Fig. 9.11, $\Delta s$ $ABC$ and $DEF$ are such that

$$\angle A = \angle D$$

$$\text{and} \quad \frac{AB}{DE} = \frac{AC}{DF}$$

*To prove* : $\triangle ABC \sim \triangle DEF$

*Proof* : Mark points $M$ and $N$ on $AB$ and $AC$ respectively such that

$$AM = DE \text{ and } AN = DF \qquad ...(i)$$

Now join $M$ to $N$.

In $\Delta s$ $DEF$ and $AMN$,

$DE = AM$ (construction)

$\angle D = \angle A$ (given)

$DF = AN$ (construction)

$\therefore$ $\triangle DEF \cong \triangle AMN$

$\therefore$ $\angle E = \angle M$ ...(*ii*) (corresponding angles)

$\angle F = \angle N$ ...(*iii*) (corresponding angles)

Now, $\dfrac{AB}{DE} = \dfrac{AC}{DF}$ (given)

$\therefore$ $\dfrac{AB}{AM} = \dfrac{AC}{AN}$ $(DE = AM, DF = AN)$

$\therefore$ $\dfrac{BM}{AM} = \dfrac{CN}{AN}$

$\therefore$ $MN \parallel BC$ (Theorem 9.2)

$\therefore$ $\angle B = \angle M$ (corresponding angles)

$= \angle E$ [given by (*ii*)]

Similarly, $\angle C = \angle F$

$\therefore$ $\triangle ABC \sim \triangle DEF$ (*AA* criterion)

Now, the various criteria for the similarity of triangles are summarised as follows :

Two triangles are similar if

1. The corresponding sides are proportional (*SSS* criterion).
2. Two pairs of corresponding angles are equal (*AA* criterion).
3. One angle of one triangle is equal to one angle of the other triangle and the sides including these angles are proportional (*SAS* criterion).

*Note* : We know that if two $\Delta s$ $ABC$ and $DEF$ are congruent, then all the sides and angles of $\triangle ABC$ are equal to the corresponding sides and angles of $\triangle DEF$. But in case, the triangles $ABC$ and $DEF$ are similar, we have angles of the one 'equal' to the angles of the other (*i.e.* $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$),

but in respect of the sides *only the ratios* of three pairs of corresponding sides are equal.

Let $\triangle ABC \sim \triangle DEF$ (see Fig. 9.12).

How to get the corresponding sides of $\triangle DEF$ and $\triangle ABC$?

Now $AB$ is a side of $\triangle ABC$. What is the corresponding point of $A$? It is $D$. Similarly, $E$ is the corresponding point of $B$. Therefore, the side corresponding to $AB$ of $\triangle ABC$ is the side $DE$ of $\triangle DEF$.

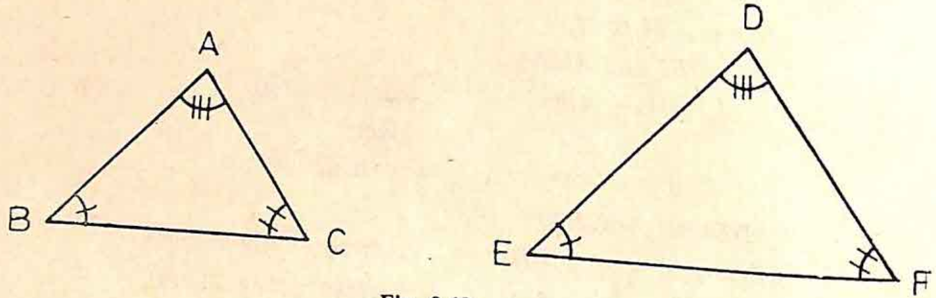



Fig. 9.12

In the same way, the other two pairs of corresponding sides are $BC$ and $EF$, $AC$ and $DF$.

In Fig. 9.13, it can be seen that $\triangle AED \sim \triangle ABC$ ($AA$ criterion). Here the pairs of corresponding sides are $AE$ and $AB$, $ED$ and $BC$, $AD$ and $AC$.

Let us solve some examples to illustrate the use of criteria that have been discussed above for two triangles to be similar.

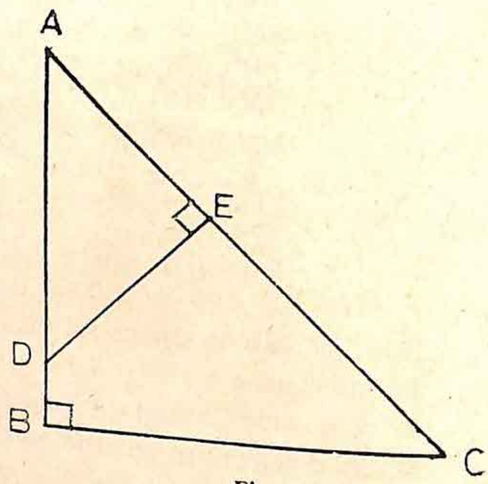



Fig. 9.13

**Example 1:** In Fig. 9.14, $AD$ is the bisector of $\angle A$ of $\triangle ABC$. Prove that

$$\frac{BA}{AC} = \frac{BD}{DC}$$

**Solution :**

*Given* : $\triangle ABC$ in which $AD$ is the bisector of $\angle A$.

*To prove* : $\frac{BA}{AC} = \frac{BD}{DC}$

*Construction* : Through $C$, draw a line parallel to $DA$ to meet $BA$ produced at $E$.

*Proof* : In $\triangle BCE$, $DA$ is parallel to $CE$.

$\therefore$ $\angle 1 = \angle 3$ (corresponding angles) (*i*)

$\angle 2 = \angle 4$ (alternate angles) ...(*ii*)

and

$\angle 1 = \angle 2$ ($AD$ is the bisector of $\angle A$) ...(*iii*)

From (*i*), (*ii*) and (*iii*), we get

$\angle 3 = \angle 4$

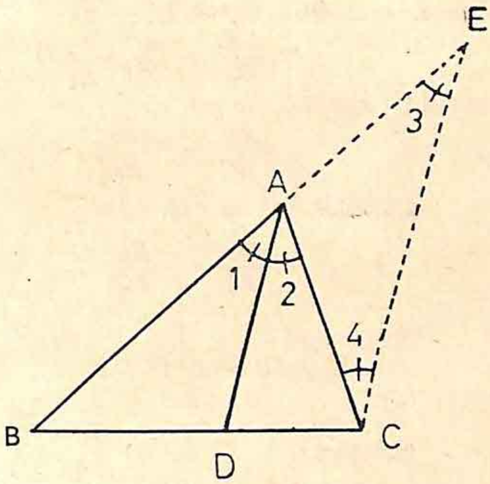



**Fig. 9.14**

Hence, $AC = AE$

In $\triangle BCE$, since $DA \parallel CE$,

$$\frac{BA}{AE} = \frac{BD}{DC}$$

$$\therefore \quad \frac{BA}{AC} = \frac{BD}{DC} \quad (\because \; AC = AE)$$

**Example 2 :** If three or more parallel lines are each cut by two transversals, the intercepted segments on the two transversals are proportional.

**Solution :**

*Given* : Three parallel lines $l_1$, $l_2$ and $l_3$ are each cut by two transversals $t_1$ and $t_2$ at points $A$, $B$, $C$ and $D$, $E$, $F$ respectively as shown in Fig. 9.15.

*To prove* :

$$\frac{AB}{BC} = \frac{DE}{F}$$

*Construction* : Join $C$ to $D$. Denote the point where $CD$ cuts the line $l_2$ by $X$.

*Proof* : In $\triangle ACD$, $BX \parallel AD$.

$$\therefore \quad \frac{CB}{BA} = \frac{CX}{XD} \quad \text{(Basic Proportionality Theorem)}$$

*i.e.* $$\frac{AB}{BC} = \frac{DX}{XC} \quad ...(i)$$

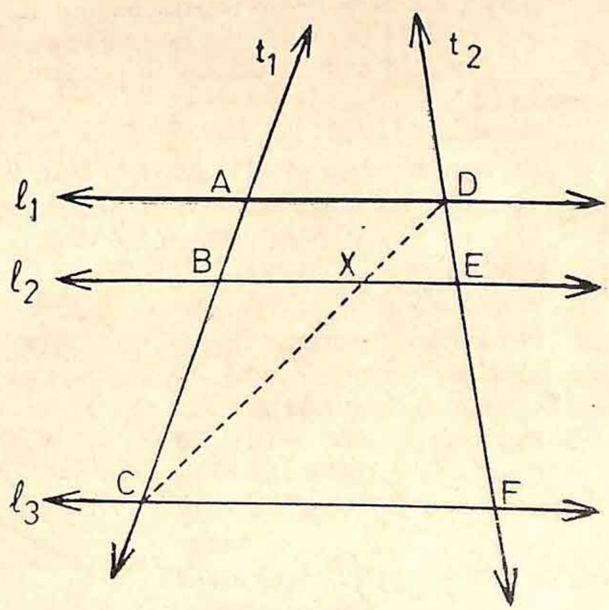



**Fig. 9.15**

Again, in $\triangle DCF$, since $XE \parallel CF$,

$$\frac{DX}{XC} = \frac{DE}{EF} \quad \ldots(ii)$$

From $(i)$ and $(ii)$, we get

$$\frac{AB}{BC} = \frac{DE}{EF}.$$

**Example 3 :** In Fig. 9.16,

$$\frac{EA}{EC} = \frac{EB}{ED}.$$

Prove that :

$(i)$ $\triangle EAB \sim \triangle ECD$

$(ii)$ $AB \parallel CD$

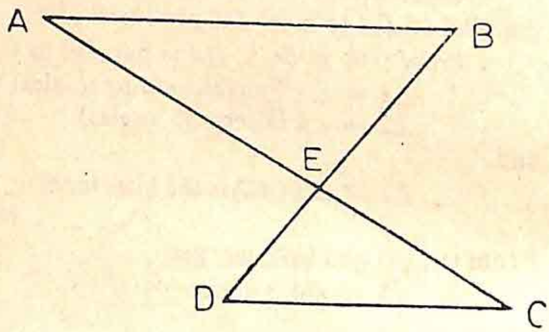



Fig. 9.16

**Solution :**

*Given* : $\dfrac{EA}{EC} = \dfrac{EB}{ED}$

*To prove* : $(i)$ $\triangle EAB \sim \triangle ECD$

and $(ii)$ $AB \parallel CD$

*Proof* : In $\triangle$s $EAB$ and $ECD$,

$\angle AEB = \angle CED$ (vertically opposite angles)

$\dfrac{EA}{EC} = \dfrac{EB}{ED}$ (given)

$\therefore$ $\triangle EAB \sim \triangle ECD$ ($SAS$ criterion)

Hence, $\angle EAB = \angle ECD$ (corresponding angles of the triangles) $\ldots(i)$

$\angle EBA = \angle CDE$ (corresponding angles of the triangles) $\ldots(ii)$

$\angle EAB$ and $\angle ECD$ are also alternate angles. These alternate angles are equal by $(i)$.

Hence, $AB \parallel CD$

**Example 4 :** In Fig. 9.17, $ABC$ and $DBC$ are two right triangles with the common hypotenuse $BC$ and with their sides $AC$ and $DB$ intersecting at $P$. Prove that

$$AP \cdot PC = DP \cdot PB$$

**Solution :**

*Given* : Two right triangles $ABC$ and $DBC$ with common hypotenuse $BC$ and with their sides $AC$ and $DB$ intersecting at $P$ (Fig. 9.17).

*To prove* : $AP \cdot PC = DP \cdot PB$

*Proof* : In $\triangle$s $APB$ and $DPC$,

$\angle BAP = \angle PDC$ (each a right angle)

and $\angle APB = \angle DPC$ (vertically opposite angles)

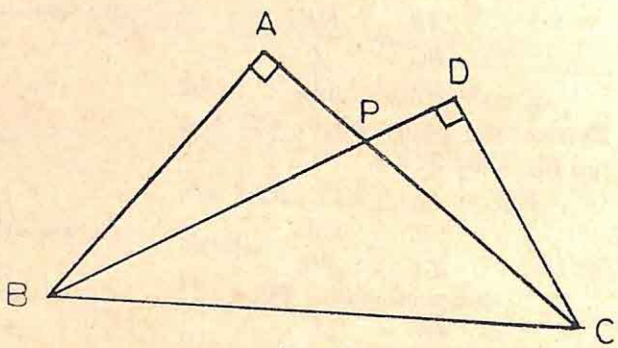



Fig. 9.17

$\therefore$ $\Delta$s $APB$ and $DPC$ are similar ($AA$ criterion)

Hence, $$\frac{AP}{DP} = \frac{PB}{PC}$$

$\therefore$ $$AP \cdot PC = DP \cdot PB$$

**Example 5 :** Prove that the ratio of the areas of two similar triangles is equal to the ratio of squares of the corresponding sides.

**Solution :**

*Given* : $\Delta\ ABC \sim \Delta\ DEF$ (Fig. 9.18).

*To prove* :

$$\frac{\text{Area}\ (\Delta\ ABC)}{\text{Area}\ (\Delta\ DEF)} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$$

*Construction* : Draw $AP \perp BC$ and $DQ \perp EF$.

*Proof* : $$\text{Area}\ (\Delta\ ABC) = \frac{1}{2}\ BC \times AP$$

and $$\text{Area}\ (\Delta\ DEF) = \frac{1}{2}\ EF \times DQ$$

$\therefore$ $$\frac{\text{Area}\ (\Delta\ ABC)}{\text{Area}\ (\Delta\ DEF)} = \frac{BC \times AP}{EF \times DQ} \qquad \ldots(i)$$

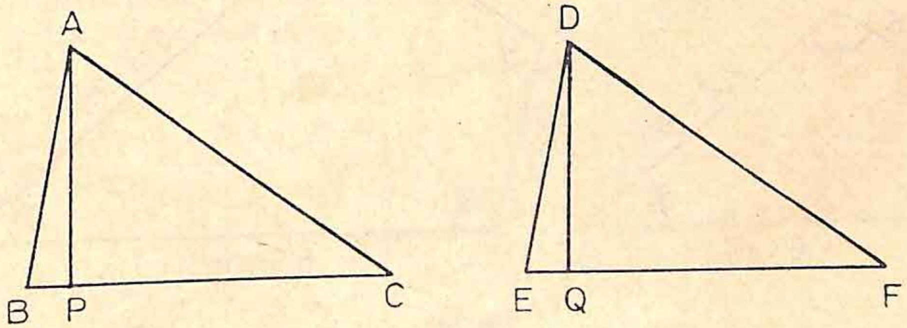



Fig. 9.18

In $\Delta$s $APB$ and $DQE$,

$\angle ABP = \angle DEQ$ (given)

$\angle APB = \angle DQE$ (each a right angle)

$\therefore$ $\Delta\ APB \sim \Delta DQE$ ($AA$ criterion)

And so $$\frac{AB}{DE} = \frac{AP}{DQ} \qquad \ldots(ii)$$

But $$\frac{AB}{DE} = \frac{BC}{EF} \quad \text{(given)} \qquad \ldots(iii)$$

Thus, from (*i*), (*ii*) and (*iii*), we obtain

$$\frac{\text{Area } (\triangle\ ABC)}{\text{Area } (\triangle\ DEF)}=\frac{BC \cdot AP}{EF \cdot DQ}=\frac{BC}{EF}\cdot\frac{AP}{DQ}=\frac{BC}{EF}\cdot\frac{BC}{EF}=\frac{BC^2}{EF^2}$$

Similarly, we can prove that

$$\frac{\text{Area } (\triangle\ ABC)}{\text{Area } (\triangle\ DEF)}=\frac{AB^2}{DE^2}=\frac{AC^2}{DF^2}$$

## Exercises 9.1

1. In each of the following, there is a pair of figures. Which pair of figures are similar and which are not similar ? Give reasons.

(*i*)

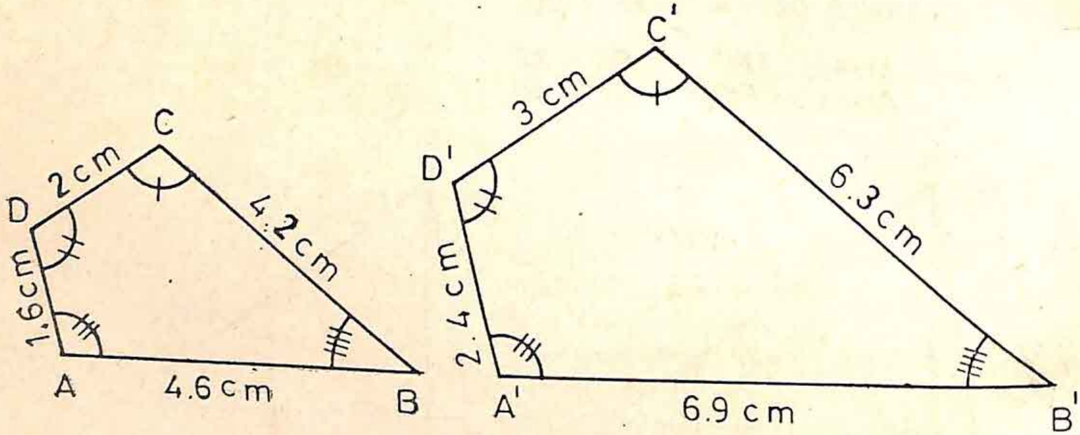



Fig. 9.19

(*ii*)

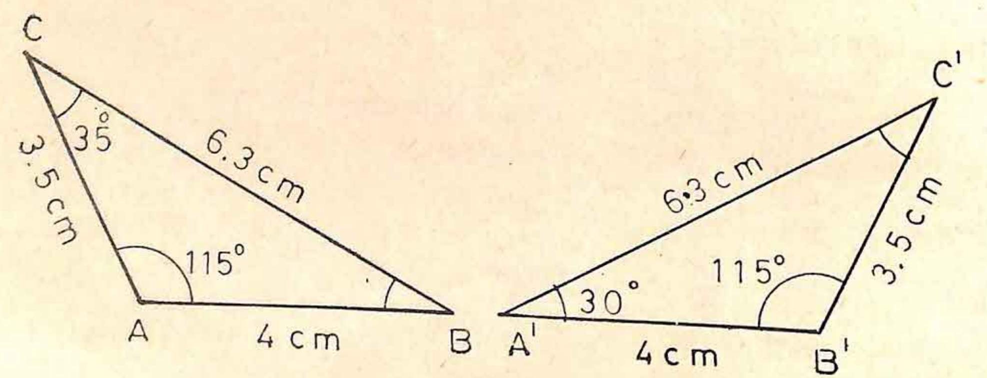



Fig. 9.20

(*iii*)

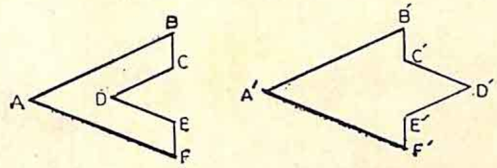



Fig. 9.21

(*iv*)

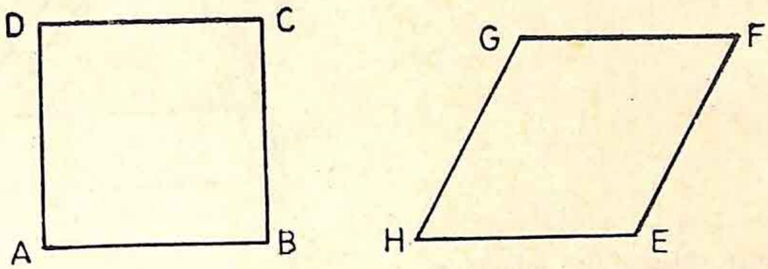



Fig. 9.22

2. In Fig. 9.23, $\triangle ABC \sim \triangle ADE$. If $AD = 5$ cm, $AE = 6$ cm, $BC = 12$ cm and $AB = 15$ cm, determine $AC$ and $DE$.

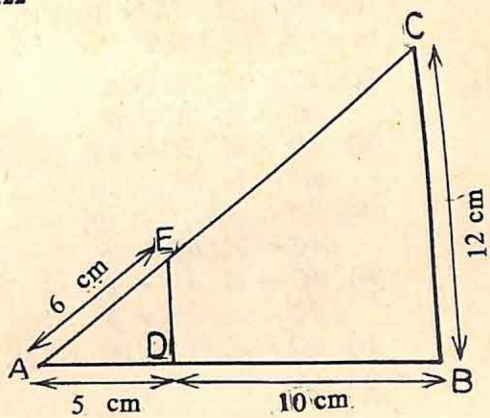



Fig. 9.23

3. In Fig. 9.24, $AC \parallel BD$. Prove that
   (*i*) $\triangle ACE \sim \triangle BDE$
   (*ii*) $\dfrac{AE}{CE} = \dfrac{BE}{DE}$

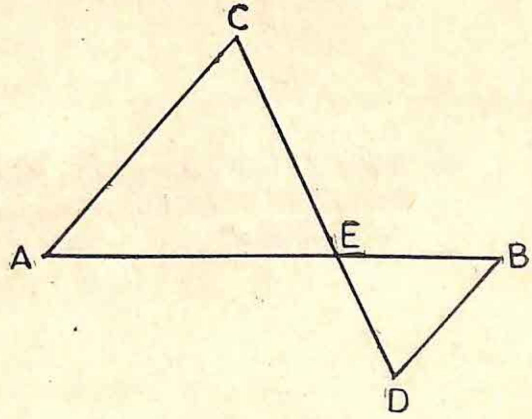



Fig. 9.24

4. In Fig. 9.25, find the values of $x$ if $DE \parallel AB$.

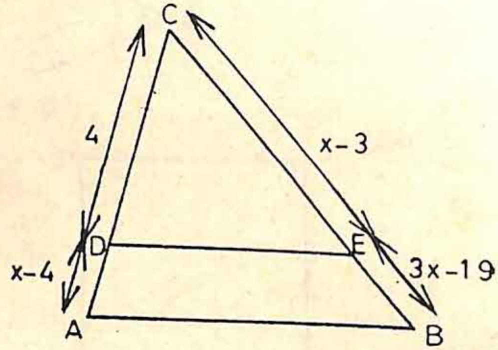



Fig. 9.25

5. Under which of the following conditions will $DE$ be parallel to $AB$ in Fig. 9.26 ?
   (i) $AC = 10$, $CD = 4$, $EC = 2$, $BC = 5$
   (ii) $AD = 6$, $EC = 14$, $BC = 18$, $DC = 21$
   (iii) $BE = 20$, $DC = 10$, $AC = 25$, $BC = 36$
   (iv) $AC = 12$, $AD = 8$, $EC = 3$, $BC = 9$
   (v) $EC = 6$, $BC = 14$, $AD = 12$, $DC = 8$

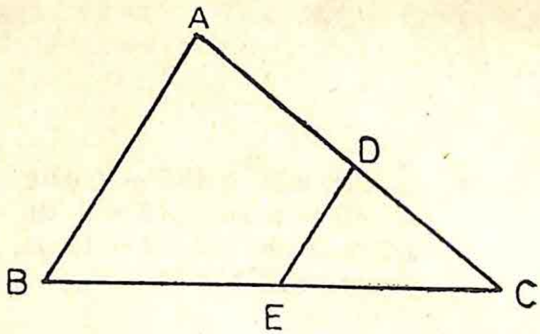
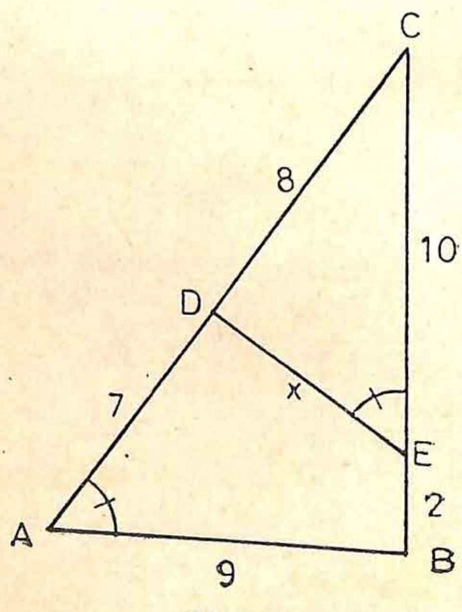



Fig. 9.26

6. In Fig. 9.27, if $\angle A = \angle CED$, prove that $\triangle CAB \sim \triangle CED$. Also, find the value of $x$.



7. $D$ is a point on the side $BC$ of $\triangle ABC$ such that $\angle ADC = \angle BAC$. Prove that $\dfrac{CA}{CD} = \dfrac{CB}{CA}$.

8. In Fig. 9.28, $DEFG$ is a square and $\angle C = 90°$. Prove that
   (i) $\triangle ADG \sim \triangle GCF$
   (ii) $\triangle ADG \sim \triangle FEB$
   (iii) $\dfrac{AD}{DG} = \dfrac{FE}{EB}$
   (iv) $DE^2 = AD \times EB$

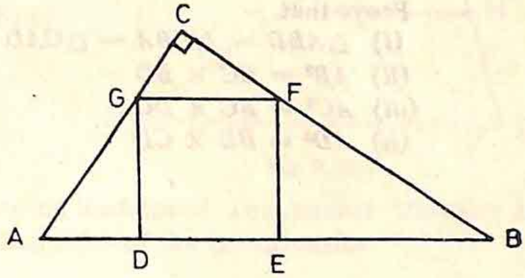



Fig. 9.28

9. In Fig. 9.29, $AD$ and $AE$ are respectively the bisectors of the interior and exterior angles at $A$. Prove that $\dfrac{BD}{BE} = \dfrac{CD}{CE}$

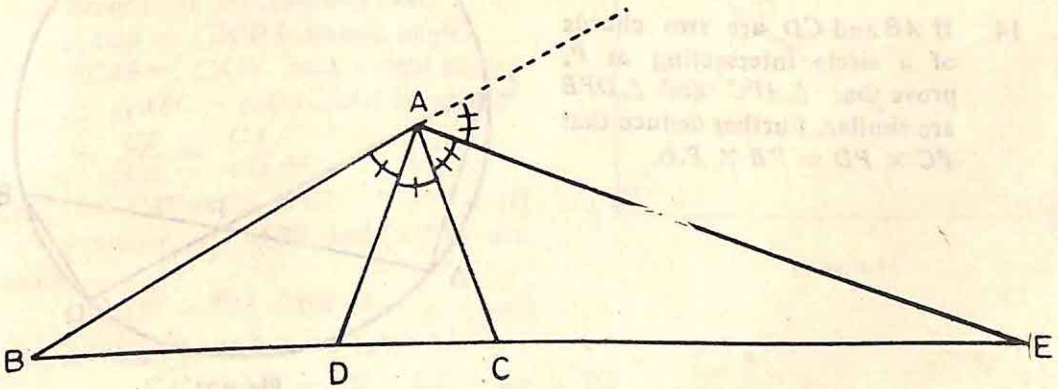



Fig. 9.29

10. A vertical stick 15 cm long casts its shadow 10 cm long on the ground. The flag pole casts a shadow 60 cm long at the same time. What is the height of the flag pole ?

11. The perimeters of two similar triangles are 24 cm and 18 cm respectively. If one side of first triangle is 8 cm, what is the corresponding side of the other triangle ?

12. $ABD$ is a triangle in which $\angle DAB = 90°$ and $AC \perp BD$. Prove that
   (i) $\triangle ABD \sim \triangle CBA \sim \triangle CAD$
   (ii) $AB^2 = BC \times BD$
   (iii) $AC^2 = BC \times DC$
   (iv) $AD^2 = BD \times CD$

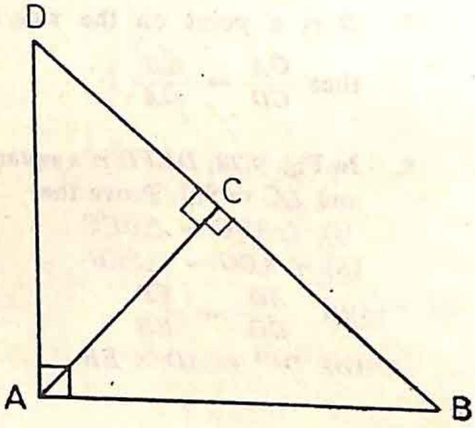



Fig. 9.30

13. Prove that the line segments joining the mid-points of the sides of a triangle form four triangles, each of which is similar to the original triangle.

14. If $AB$ and $CD$ are two chords of a circle intersecting at $P$, prove that $\triangle APC$ and $\triangle DPB$ are similar. Further deduce that $PC \times PD = PB \times PA$.

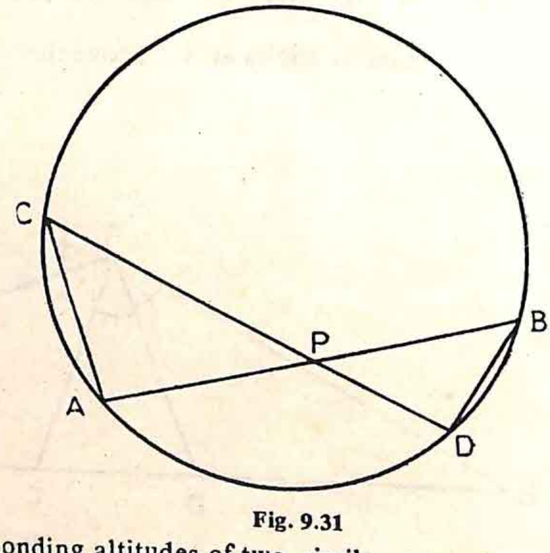



Fig. 9.31

15. Prove that the ratio of the corresponding altitudes of two similar triangles is equal to the ratio of their corresponding sides.

16. In Fig. 9.32, $ABCD$ is a trapezium with $AB \parallel DC$. If $\triangle AED \sim \triangle BEC$, prove that $AD = BC$.

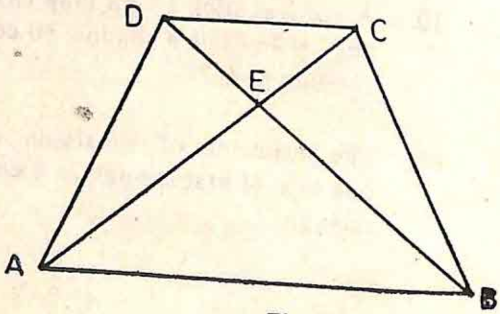



Fig. 9.32

17. In $\triangle PQR$ (Fig. 9.33), $G$ is the mid-point of $PR$ and $H$ is the mid-point of $QR$. What is the ratio of area ($\triangle GHR$) to area ($\triangle PQR$)?

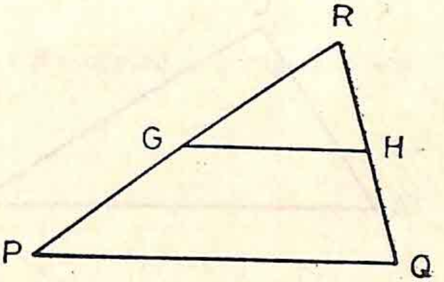


Fig. 9.33

18. Prove that the ratio of the corresponding medians of two similar triangles is equal to the ratio of the corresponding sides of the two triangles.

## 9.3 Pythagoras Theorem

**Theorem : In a right triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.**

*Given* : In $\triangle ABC$, $\angle A=90°$ (Fig. 9.34).

*To prove* : $CB^2 = CA^2 + AB^2$

*Construction* : Draw $AD \perp BC$

*Proof* : In $\triangle s$ $ABC$ and $DAC$,

$\angle ACB = \angle ACD$ (common angle)

$\angle CAB = \angle ADC$ (each a right angle)

$\therefore \triangle ABC \sim \triangle DAC$ ($AA$ criterion)

$\therefore \dfrac{CB}{CA} = \dfrac{CA}{CD}$

*i.e.* $CA^2 = CB \times CD$ ...(*i*)

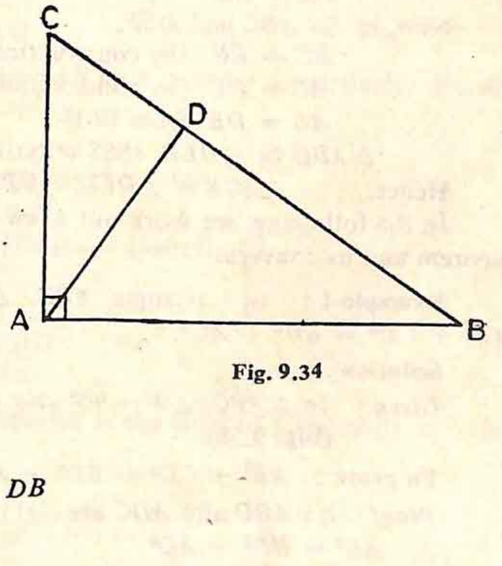


Fig. 9.34

Similarly, $\triangle s$ $ABC$ and $DBA$ are similar.

$\therefore AB^2 = CB \times DB$ ...(*ii*)

Adding (*i*) and (*ii*), we have

$$CA^2 + AB^2 = CB \times CD + CB \times DB$$
$$= CB \times (CD + DB)$$
$$= CB \times CB$$
$$= CB^2$$

(This theorem is also known as Bodhayan theorem.)

We now prove the converse of the Pythagoras theorem.

**Theorem : In a triangle, if the square on one side is equal to the sum of the squares on the remaining two sides, then the angle opposite to the first side is a right angle.**

*Given* : In $\triangle ABC$, $AB^2 = BC^2 + CA^2$ (Fig 9.35).

*To prove* : $\angle ACB = 90°$

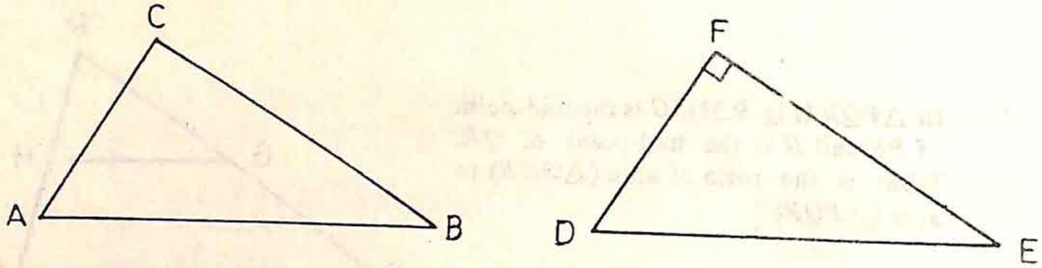


Fig. 9.35

*Construction* : Construct a right triangle $DEF$ such that $DF = AC$, $EF = BC$ and $\angle EFD = 90°$.

*Proof* : In $\triangle DEF$, $\angle EFD = 90°$ (by construction)

$\therefore \quad DE^2 = EF^2 + DF^2$ ...(*i*) (by Pythagoras theorem)

$= BC^2 + AC^2$

$AB^2 = BC^2 + AC^2$ ...(*ii*) (given)

$\therefore \quad DE^2 = AB^2$ [from (*i*) and (*ii*)]

*i.e.* $\quad DE = AB$ ...(*iii*)

Now, in $\triangle s$ $ABC$ and $DEF$,

$BC = EF$ (by construction)

$AC = DF$ (by construction)

$AB = DE$ [from (*iii*)]

$\triangle ABC \cong \triangle DEF$ (SSS criterion)

Hence, $\quad \angle ACB = \angle DFE = 90°$

In the following, we work out a few examples to illustrate the use of Pythagoras theorem and its converse.

**Example 1 :** In triangle $ABC$, $\angle A = 90°$. If $AD \perp BC$, then prove that $AB^2 + CD^2 = BD^2 + AC^2$.

**Solution :**

*Given* : In $\triangle ABC$, $\angle A = 90°$ and $AD \perp BC$ (Fig. 9.36).

*To prove* : $AB^2 + CD^2 = BD^2 + AC^2$

*Proof* : $\triangle s$ $ABD$ and $ADC$ are right triangles.

$\therefore \quad AB^2 = BD^2 + AD^2$ ...(*i*)

and $\quad AC^2 = AD^2 + CD^2$ ...(*ii*)

From (*i*) and (*ii*), we have

$AD^2 = AB^2 - BD^2 = AC^2 - CD^2$

*i.e.* $\quad AB^2 + CD^2 = BD^2 + AC^2$

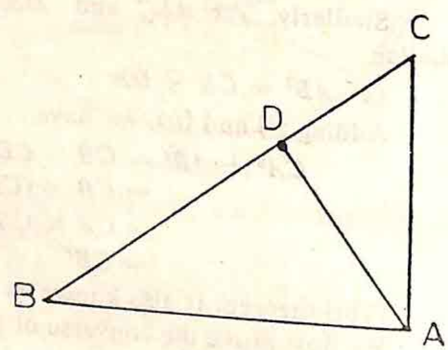


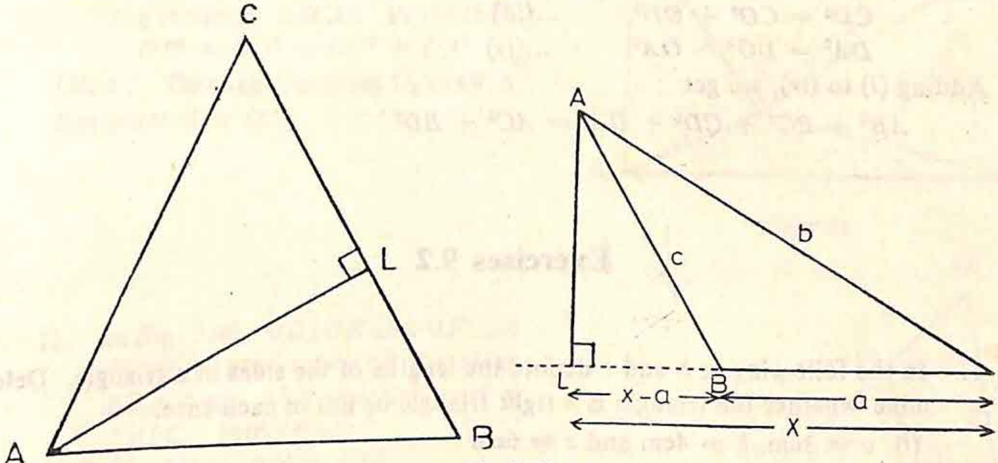
Fig. 9.36

**Example 2 :** In triangle $ABC$, $\angle C < 90°$ and $AL$ is perpendicular to $BC$. Prove that $AB^2 = AC^2 + BC^2 - 2BC \times CL$.

**Solution :**

*Given* : In $\triangle ABC, \angle C < 90°$ and $AL \perp BC$ [or $CB$ produced as in Fig. 9.37 (*ii*)]

*To prove* : $AB^2 = AC^2 + BC^2 - 2BC \times CL$




(*i*) **Fig. 9.37** (*ii*)

*Proof* : Let us denote the sides $BC$, $CA$ and $AB$ by $a$, $b$ and $c$ respectively. Further, let $x$ and $d$ denote the sides $CL$ and $AL$ respectively.

Now, $ALC$ is a right triangle.

$\therefore \quad b^2 = d^2 + x^2 \quad (i)$ (by Pythagoras theorem)

Also, $ALB$ is a right triangle.

$\therefore \quad c^2 = d^2 + (a - x)^2$ (by Pythagoras theorem)

$= d^2 + a^2 + x^2 - 2ax$

$= (d^2 + x^2) + a^2 - 2ax$

$= b^2 + a^2 - 2ax$ [using (*i*)]

$\therefore \quad AB^2 = AC^2 + BC^2 - 2BC \times CL$

**Example 3 :** Prove that the sum of the squares of the sides of a rhombus is equal to the sum of the squares of its diagonals.

**Solution :**

*Given* : $ABCD$ is a rhombus with diagonals $AC$ and $BD$ (Fig. 9.38).

*To prove* :

$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$

*Proof* : Since diagonals of a rhombus bisect at right angles, we have

$AO = OC$, $BO = OD$,

$\angle AOB = \angle BOC = \angle COD$

$= \angle DOA = 90°$

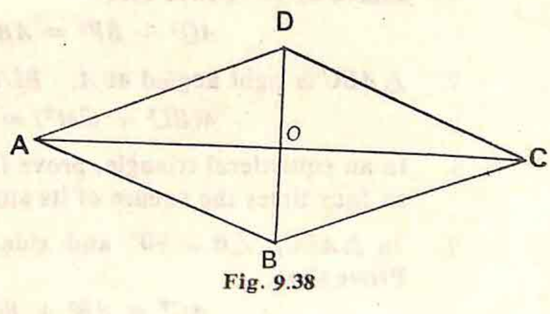



**Fig. 9.38**

In $\triangle AOB$, $\angle AOB = 90°$

$\therefore \quad AB^2 = AO^2 + OB^2 \quad ...(i)$ (by Pythagoras theorem)

Similarly, $BC^2 = BO^2 + OC^2 \quad ...(ii)$

$CD^2 = CO^2 + OD^2 \quad ...(iii)$

$DA^2 = DO^2 + OA^2 \quad ...(iv)$

Adding (*i*) to (*iv*), we get

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$$

## Exercises 9.2

1. In the following $a$, $b$ and $c$ denote the lengths of the sides of a triangle. Determine whether the triangle is a right triangle or not in each case.
   (*i*) $a = 3$cm, $b = 4$cm and $c = 6$cm
   (*ii*) $a = 7$cm, $b = 12$cm and $c = 5$cm
   (*iii*) $a = 12$cm, $b = 16$cm and $c = 20$cm
2. $ABC$ is an equilateral triangle with side $2a$ and $AD \perp BC$. Show that
   (*i*) $AD = a\sqrt{3}$
   (*ii*) area $(\triangle ABC) = a^2\sqrt{3}$
3. A ladder 20m long reaches a window of a house 16m above the ground. Determine the distance of the foot of the ladder from the house.
4. Two poles of heights 6m and 11m stand on a plane ground. If the distance between their feet is 12m, find the distance between their tops.
5. A man goes 10m due East and then 30m due North. Find his distance from the starting point.
6. $P$ and $Q$ are points on the sides $CA$ and $CB$ respectively of a $\triangle ABC$, right angled at $C$. Prove that
   $$AQ^2 + BP^2 = AB^2 + PQ^2$$
7. $\triangle ABC$ is right angled at $A$. $BL$ and $CM$ are its medians. Prove that
   $$4(BL^2 + CM^2) = 5BC^2$$
8. In an equilateral triangle, prove that three times the square of one side is equal to four times the square of its altitude.
9. In $\triangle ABC$, $\angle B > 90°$ and side $CB$ is produced to $D$ such that $AD \perp CD$. Prove that
   $$AC^2 = AB^2 + BC^2 + 2BC \times BD$$

10. In Fig. 9.39, $O$ is any point inside the rectangle $ABCD$. Prove that

$$OB^2 + OD^2 = OC^2 + OA^2$$

[*Hint* : Through the point $O$, draw a line parallel to $BC$.]

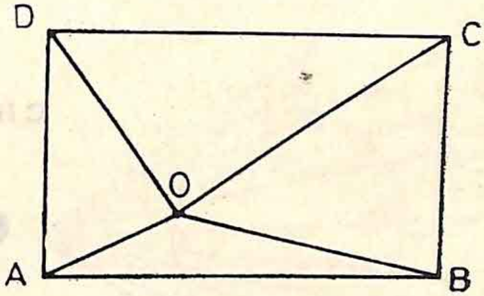



Fig. 9.39

11. In Fig. 9.40, $OD$, $OE$ and $OF$ are drawn perpendiculars to the sides $BC$, $CA$ and $AB$ respectively of a $\triangle ABC$. Prove that :

(*i*) $AF^2 + BD^2 + CE^2 = OA^2 + OB^2 + OC^2 - OD^2 - OE^2 - OF^2$

(*ii*) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$

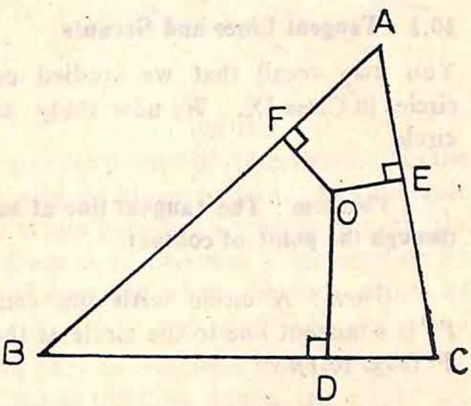



Fig. 9.40

12. $ABC$ is an isosceles triangle, right angled at $C$. Prove that

$$AB^2 = 2AC^2$$

CHAPTER 10

# Circles

## 10.1 Tangent Lines and Secants

You may recall that we studied certain basic definitions and properties concerning circles in Class IX. We now study some more important results connected with the circle.

**Theorem : The tangent line at any point of a circle is perpendicular to the radius through the point of contact.**

*Given* : A circle with the centre $O$. $PT$ is a tangent line to the circle at the point $P$ (Fig. 10.1).

*To prove* : $OP \perp PT$

*Construction* : Let $Q$ be any point, other than $P$, on $PT$. Join $OQ$.

*Proof* : Since $Q$ lies in the exterior of circle,

$\therefore \; OQ > OP$

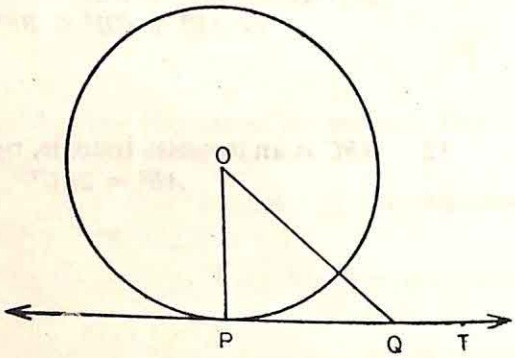



**Fig. 10.1**

Thus, of all the segments that can be drawn from the centre $O$ to any point on the line $PT$, $OP$ is the shortest.

Now, we know that the shortest segment that can be drawn from a given point to a given line is the perpendicular from the given point to the given line. Hence, $OP \perp PT$.

The converse of the above theorem is also true. So we have :

**The line through a point on a circle perpendicular to the radius through the point is the tangent line to the circle at that point.**

### Tangent line as a Limiting case of a Secant

We may recall the distinction between a secant and a tangent line which was discussed in Class IX. Let $P$ be a point exterior to the circle (Fig. 10.2).

Draw secants to the circle passing through $P$.

How many secants can be drawn to the circle passing through $P$ ? Obviously as many as we like. Now how many tangent lines can be drawn to the circle passing through $P$ ?

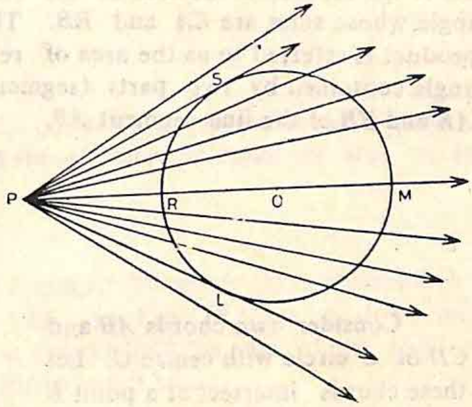


**Fig. 10.2**

Draw a secant $POM$ passing through $P$ and the centre $O$ of the circle (Fig. 10.2). If this secant through $P$ is rotated about $P$ in either direction, we observe that the distance between the points of intersection of the secant with the circle goes on decreasing and ultimately it reduces to zero. In this case we say that two points of intersection coincide and secant becomes a tangent line to the circle. If the tangent line is further moved away from $O$, it becomes a line exterior to the circle. **Thus, a tangent line is the limiting case of a secant when the two points of intersection of the secant and the circle coincide.**

From Fig. 10.2, you can see that there are only two tangent lines $PS$ and $PL$, passing through the point $P$. The points $S$ and $L$, where the lines touch the circle are called the **points of contact.**

It can be verified by actual measurement that $PS = PL$. This fact can be verified by taking $P$ at different points and measuring $PS$ and $PL$ in each case. It can also be proved easily. See Fig. 10.3. We will prove that $PS = PL$.

In $\triangle s$ $POS$ and $POL$,

$OS = OL$ (radii of the circle)

$OP = OP$

$\angle PSO = \angle PLO$ (each a right angle)

$\therefore \triangle POS \cong \triangle POL$

$\therefore PS = PL$

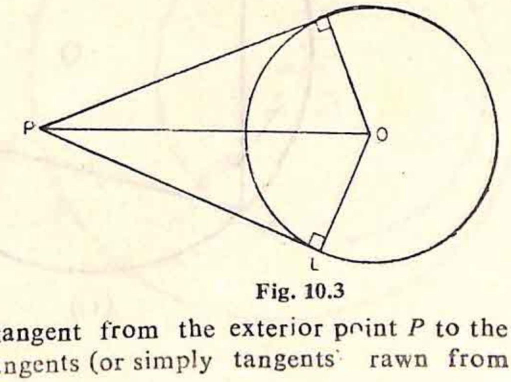


**Fig. 10.3**

The length $PS$ or $PL$ is called the length of the tangent from the exterior point $P$ to the circle. Hence, we say that the lengths of the tangents (or simply tangents) drawn from an external point to a circle are equal.

Consider a line segment $AB$ and a point $E$ on it (Fig. 10.4). Then the product $AE \times EB$ represents the area of the rectangle whose sides are $EA$ and $EB$. This product is referred to as the area of rectangle contained by two parts (segments) $AE$ and $EB$ of the line segment $AB$.

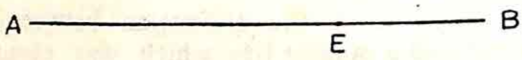



Fig. 10.4

Consider two chords $AB$ and $CD$ of a circle with centre $O$. Let these chords intersect at a point $E$ either inside or outside the circle as shown in Fig. 10.5. Join $A$ to $C$ and $B$ to $D$.

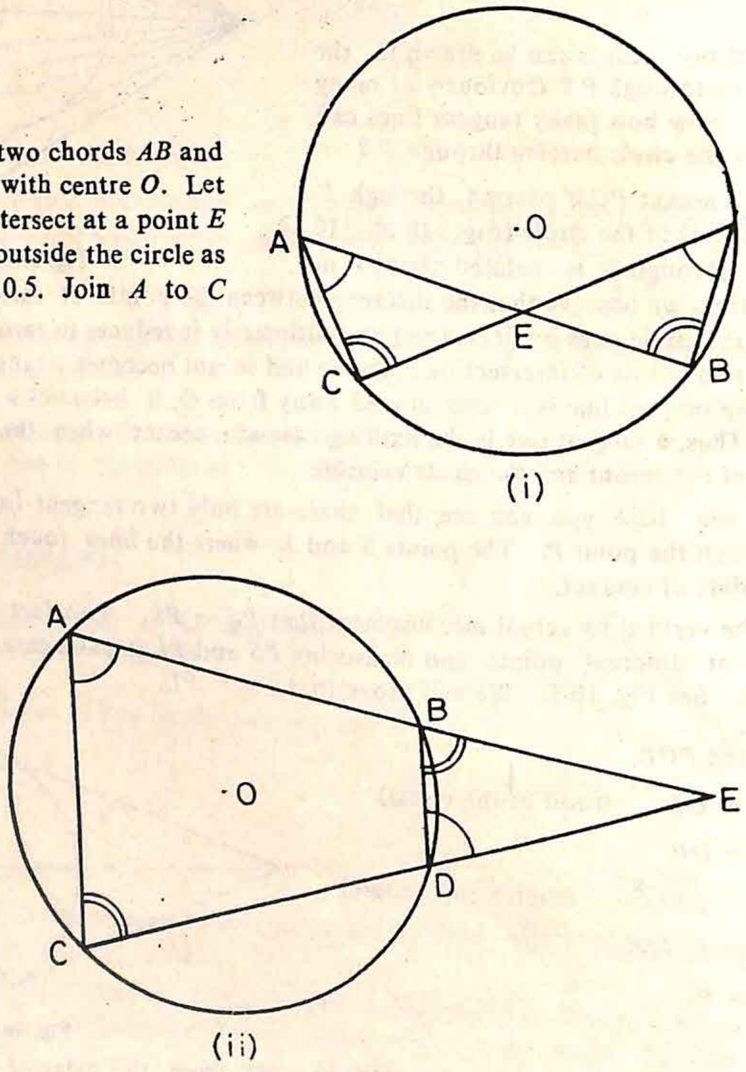



Fig. 10.5

In the figure equal angles are marked. Give reasons why these angles are equal.

$$\triangle AEC \sim \triangle DEB \quad (AA \text{ criterion})$$

$$\therefore \qquad \frac{AE}{DE} = \frac{EC}{EB}$$

Hence, $\qquad AE \times EB = DE \times EC$

**This leads to the conclusion that if two chords of a circle intersect inside or outside a circle, the rectangle contained by the segments of the one chord is equal in area to the rectangle contained by the segments of the other.**

## 10.2 Circles and Common Tangents

Given two circles with centres $A$ and $B$ and radii $R$ and $r$. There are three possibilities :

(*a*) The circles do not intersect as in Fig. 10.6. In Fig. 10.6 (*i*), it is obvious that the distance between the centres is greater than the sum of the radii of the circles. But in Figs. 10.6 (*ii*) and (*iii*) it is not so.

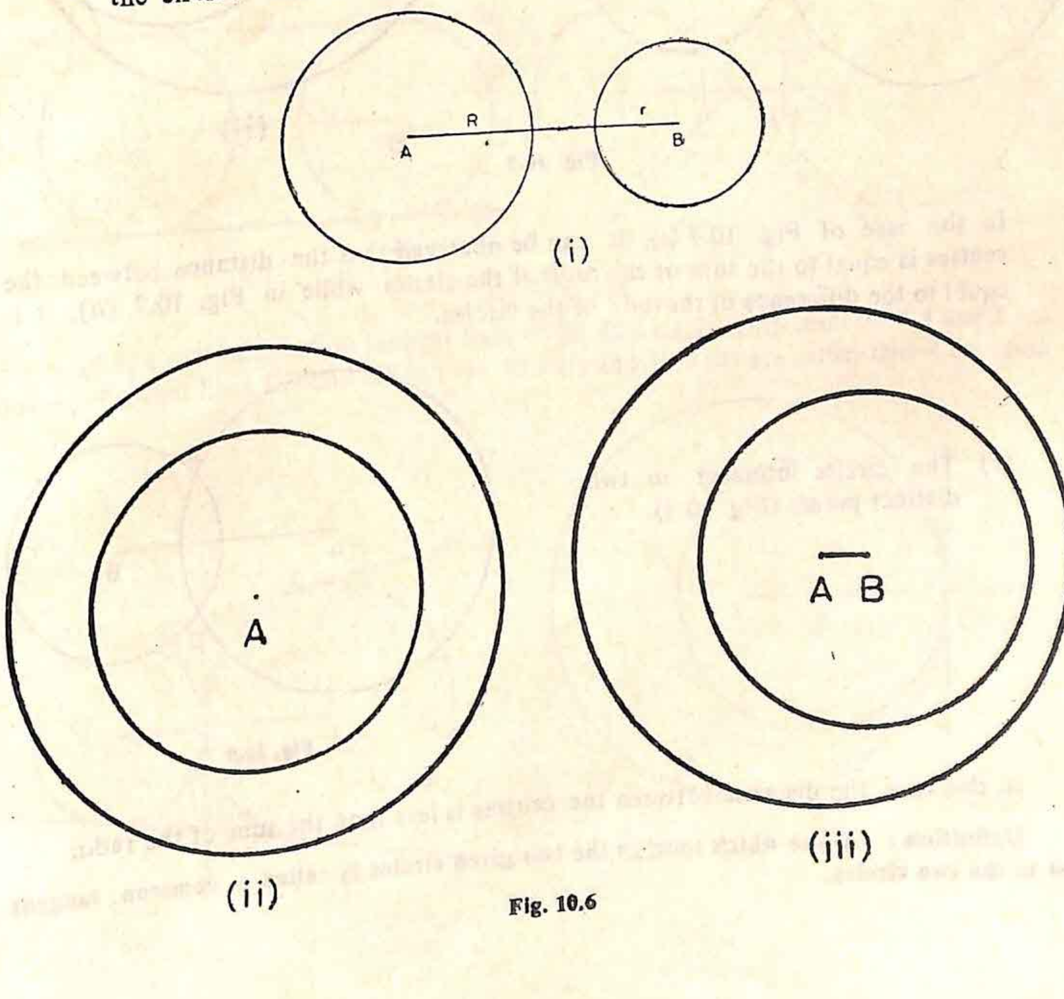
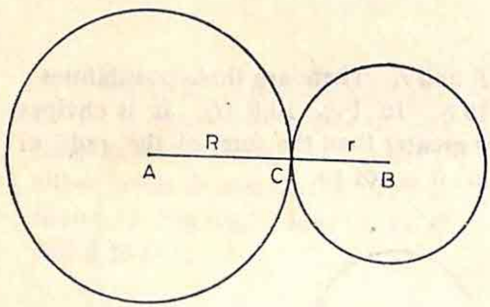
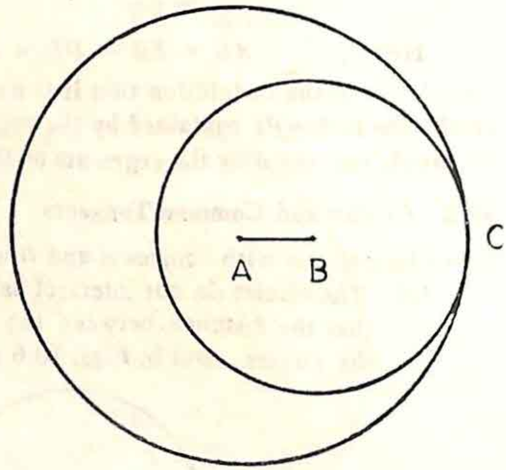



**Fig. 10.6**

(*b*) The circles touch each other externally as in Fig. 10.7 (*i*) or internally as in Fig. 10.7 (*ii*).




(i)




(ii)

Fig. 10.7

In the case of Fig. 10.7 (*i*), it can be observed that the distance between the centres is equal to the sum of the radii of the circles while in Fig. 10.7 (*ii*), it is equal to the difference of the radii of the circles.

(*c*) The circles intersect in two distinct points (Fig. 10.8).

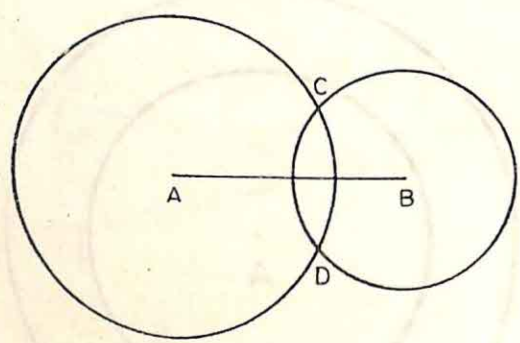
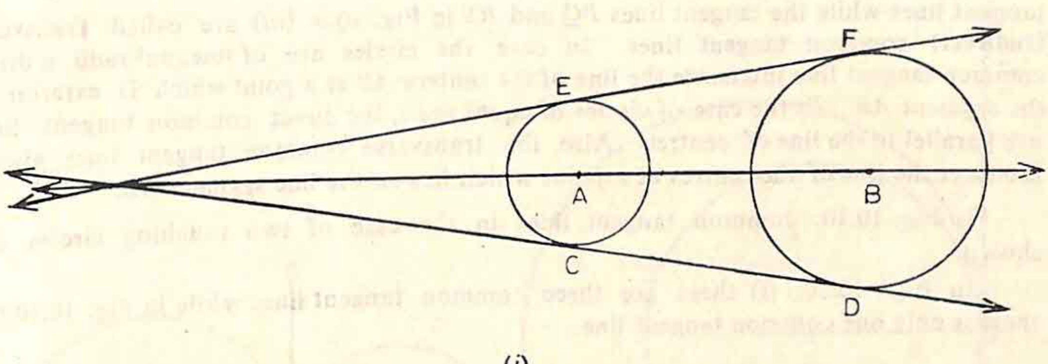
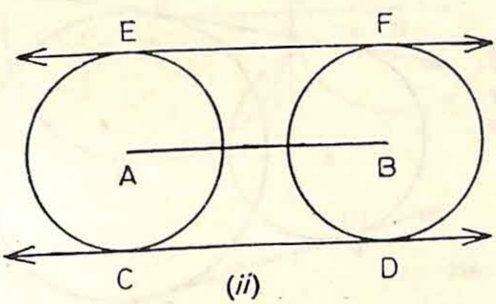



Fig. 10.8

In this case, the distance between the centres is less than the sum of the radii.

**Definition : A line which touches the two given circles is called a common tangent line to the two circles.**



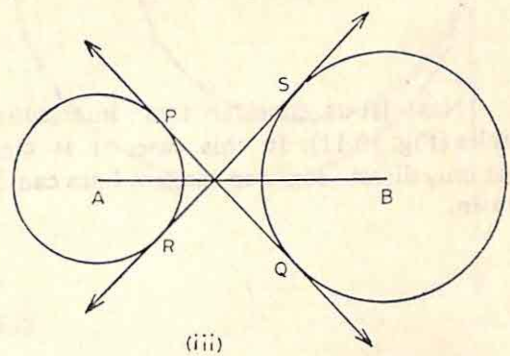
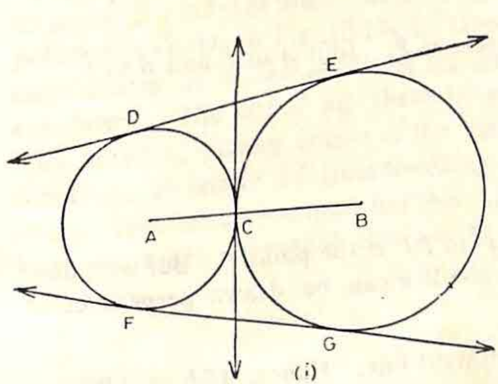




**Fig. 10.9**

In Fig. 10.9, a pair of common tangent lines to the two circles with centres at *A* and *B* are shown. Tangent lines *CD* and *EF* in Figs. 10.9 (*i*) and 10.9 (*ii*) are called **Direct** common



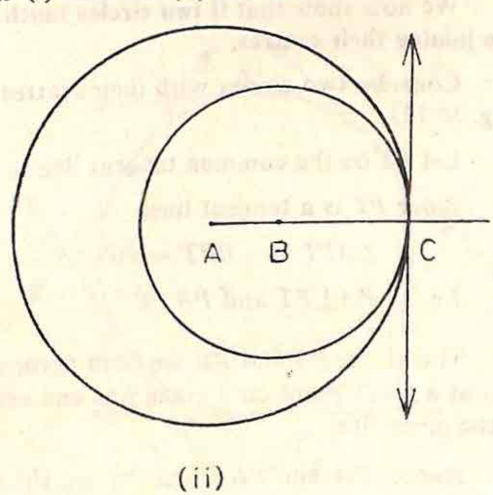




**Fig. 10.10**

tangent lines while the tangent lines $PQ$ and $RS$ in Fig. 10.9 (*iii*) are called **Transverse (Indirect)** common tangent lines. In case the circles are of unequal radii, a direct common tangent line intersects the line of the centres $AB$ at a point which is exterior to the segment $AB$. In the case of circles of equal radii, the direct common tangent lines are parallel to the line of centres. Also the transverse common tangent lines always intersect the line of the centres at a point which lies on the line segment $AB$.

In Fig. 10.10, common tangent lines in the case of two touching circles are shown.

In Fig. 10.10 (*i*) there are three common tangent lines while in Fig. 10.10 (*ii*) there is only one common tangent line.

Next let us consider two intersecting circles (Fig. 10.11). In this case, it is clear that only direct common tangent lines can be drawn.

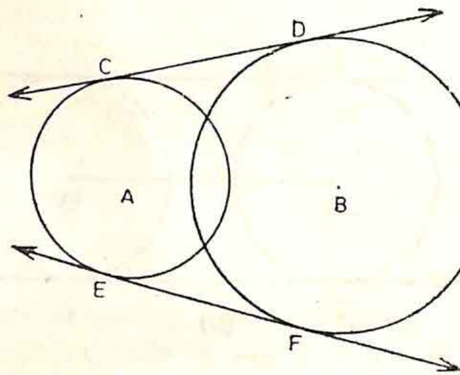



**Fig. 10.11**

We now show that **if two circles touch each other, the point of contact lies on the line joining their centres.**

Consider two circles with their centres $A$ and $B$ touching each other at the point $P$ (Fig. 10.12).

Let $PT$ be the common tangent line to the circle at $P$. Join $A$ to $P$ and $B$ to $P$.

Since $PT$ is a tangent line,

$$\angle APT = \angle BPT = 90^\circ$$

*i.e.* $PA \perp PT$ and $PB \perp PT$

Therefore, $PA$ and $PB$ are both perpendiculars to $PT$ at the point $P$. But we know that at a given point on a given line one and only one line can be drawn perpendicular to the given line.

Hence, $PA$ and $PB$ must be in the same straight line. Hence, $APB$ or $ABP$ is a straight line.

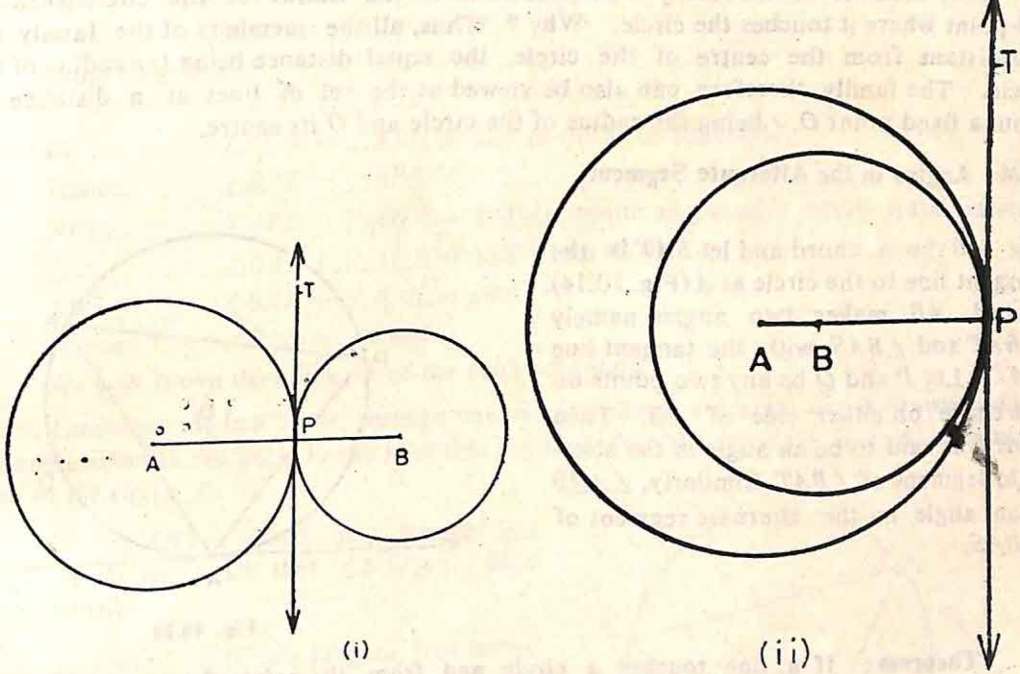



Fig. 10.12

Hence, the point of contact of two circles always lies on the line joining their centres.

## 10.3 Family of Lines Touching a Circle

Consider the tangent lines drawn at various points of a circle. In Fig. 10.13, the tangent lines drawn at a few points of the circle are shown. The set of all these tangent lines drawn at various points of the circle constitute a family of lines touching the circle. How many members has this set ? You can easily see that this set is an infinite set.

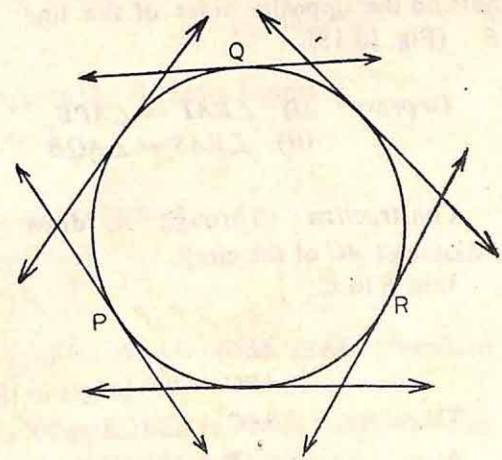



Fig. 10.13

Any member of this family is perpendicular to the radius of the circle through the point where it touches the circle. Why ? Thus, all the members of the family are equidistant from the centre of the circle, the equal distance being the radius of the circle. The family, therefore, can also be viewed as the set of lines at a distance '$r$' from a fixed point $O$, $r$ being the radius of the circle and $O$ its centre.

## 10.4 Angles in the Alternate Segment

Let $AB$ be a chord and let $SAT$ be the tangent line to the circle at $A$ (Fig. 10.14). Chord $AB$ makes two angles namely $\angle BAT$ and $\angle BAS$ with the tangent line $SAT$. Let $P$ and $Q$ be any two points on the circle on either side of $AB$. Then $\angle APB$ is said to be an angle in the alternate segment of $\angle BAT$. Similarly, $\angle AQB$ is an angle in the alternate segment of $\angle BAS$.

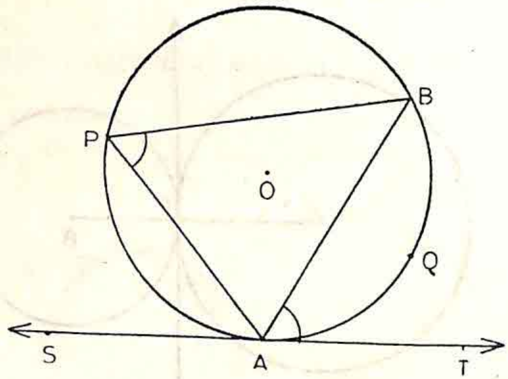



Fig. 10.14

**Theorem : If a line touches a circle and from the point of contact, a chord is drawn, the angles which this chord makes with the given line are equal respectively to the angles formed in the corresponding alternate segments.**

*Given* : A circle with centre $O$. $SAT$ is the tangent line to the circle at $A$. $AB$ is a chord through $A$. $P$ and $Q$ are points on the opposite sides of the line $AB$ (Fig. 10.15).

*To prove* : (*i*) $\angle BAT = \angle APB$
(*ii*) $\angle BAS = \angle AQB$

*Construction* : Through $A$, draw the diameter $AC$ of the circle.
Join $B$ to $C$.

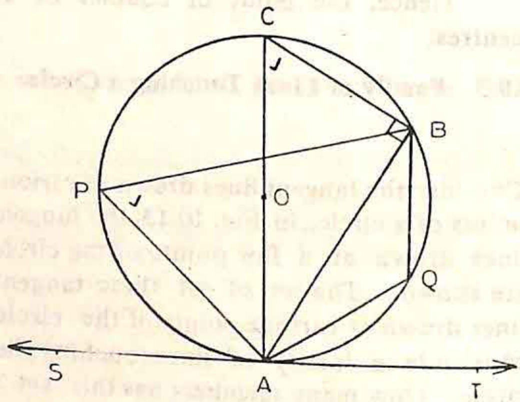



Fig. 10.15

*Proof* : In $\triangle$ $ABC$,

$\angle ABC = 90°$ (angle in the semicircle)

Therefore, $\angle BAC + \angle ACB = 90°$ ...(*i*)

Also, $\angle CAT = 90°$ (diameter is perpendicular to tangent line)

But $\angle CAT = \angle BAC + \angle BAT$

or $\angle BAC + \angle BAT = 90°$ ...(ii)

From (i) and (ii),

$\angle BAT = \angle ACB$

But, $\angle ACB = \angle APB$ (angles in the same segment)

Hence, $\angle BAT = \angle APB$ ...(iii)

Now, $\angle APB + \angle AQB = 180°$ (opposite angles of a cyclic quadrilateral)

$\therefore$ $\angle BAT + \angle AQB = 180°$ [from (iii)]

Also, $\angle BAT + \angle BAS = 180°$

Hence, $\angle BAS = \angle AQB$

We now prove the converse of the above theorem :

**Theorem : If in a circle, through one end of a chord, a straight line is drawn making an angle equal to the angle in the alternate segment, then the straight line is a tangent line to the circle.**

*Given* : $AB$ is a chord and a straight line $SAT$ is drawn such that $\angle BAT = \angle BCA$ (Fig. 10.16).

*To prove* : $SAT$ is a tangent line to the circle at $A$.

*Proof* : If $SAT$ is not a tangent line, draw $S'AT'$ a tangent line at $A$.

Since, $S'AT'$ is a tangent line and $AB$ is a chord,

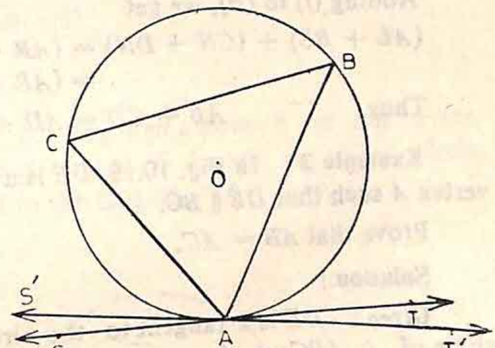


**Fig. 10.16**

$\therefore$ $\angle BAT' = \angle BCA$ (angle in the alternate segment)

But $\angle BAT = \angle BCA$ (given)

$\therefore$ $\angle BAT = \angle BAT'$

which is possible only when $SAT$ and $S'AT'$ coincide.

Hence, $SAT$ is a tangent line to the circle at $A$.

## 10.5 Applications

In the following, we solve some problems as applications of the theorems discussed in this chapter.

**Example 1** : If the sides of a quadrilateral touch a circle, prove that the sum of a pair of opposite sides is equal to the sum of the sides of the other pair.

**Solution :**

*Given :* $ABCD$ is a quadrilateral touching a circle at $L$, $M$, $N$ and $R$ (Fig. 10.17).

*To prove :* $AB + CD = BC + AD$

*Proof :* We know that the lengths of the tangents from an external point to a given circle are equal. Therefore,

$$AL = AR \quad \ldots(i)$$
$$BL = BM \quad \ldots(ii)$$
$$CN = CM \quad \ldots(iii)$$
$$DN = DR \quad \ldots(iv)$$

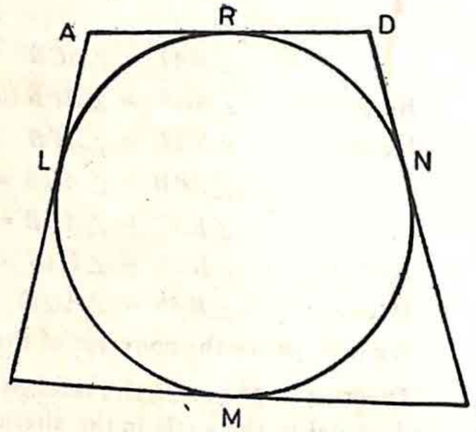


Fig. 10.17

Adding (*i*) to (*iv*), we get

$$(AL + BL) + (CN + DN) = (AR + BM) + (CM + DR)$$
$$= (AR + DR) + (BM + CM)$$

Thus, $AB + CD = AD + BC$

**Example 2 :** In Fig. 10.18, $DE$ is a tangent to the circumcircle of $\Delta\ ABC$ at the vertex $A$ such that $DE \parallel BC$.

Prove that $AB = AC$.

**Solution :**

*Given :* $DE$ is a tangent to the circumcircle of $\Delta\ ABC$ at the vertex $A$ such that $DE \parallel BC$ (Fig. 10.18).

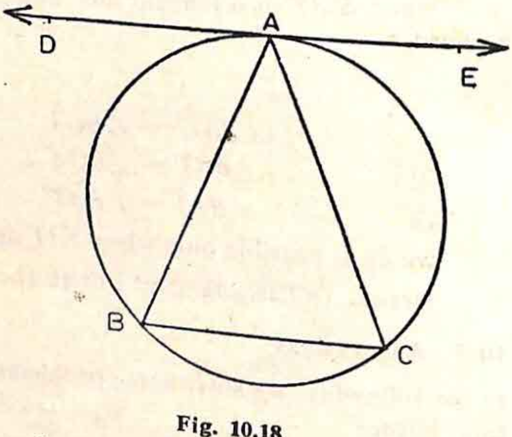


Fig. 10.18

*To prove :* $AB = AC$

*Proof :* $DE \parallel BC$ (given)

$\therefore$ $\angle DAB = \angle ABC$ (alternate angles)

But $\angle DAB = \angle ACB$ (angle in alternate segment)

$\therefore$ $\angle ABC = \angle ACB$

$\therefore$ $AB = AC$ (sides opposite to equal angle of a triangle)

**Example 3 :** $CD$ is a transverse common tangent line to circles with centres $A$ and $B$ touching them at $C$ and $D$ respectively. Prove that $\frac{AC}{BD} = \frac{AE}{BE}$.

**Solution :**

*Given :* $CD$ is a common tangent to the two circles with centres $A$ and $B$ touching them at $C$ and $D$ (Fig. 10.19).

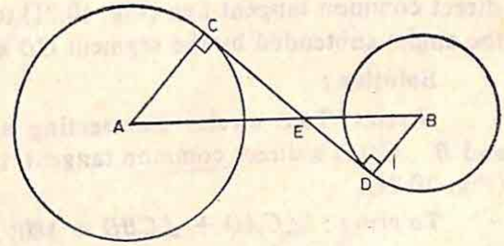



**Fig. 10.19**

*To prove :* $\frac{AC}{BD} = \frac{AE}{BE}$

*Proof :* In $\triangle$s $ACE$ and $BDE$,

$\angle ACE = \angle BDE$ (each a right angle)

$\angle AEC = \angle BED$ (vertically opposite angles)

$\therefore$ $\triangle ACE \sim \triangle BDE$ (*AA* criterion)

Hence, $\frac{AC}{BD} = \frac{AE}{BE}$

**Example 4 :** Two circles intersect at $A$ and $B$. Through a point $P$ on one of the circles, secants $PAC$ and $PBD$ are drawn such that $C$ and $D$ lie on the other circle (Fig. 10.20). Show that the chord $CD$ is parallel to the tangent $PM$.

**Solution :**

*Given :* Two circles intersect at $A$ and $B$. Through a point $P$, secants $PAC$ and $PBD$ are drawn such that $C$ and $D$ lie on the other circle. $C$ and $D$ are joined. $PM$ is a tangent at $P$.

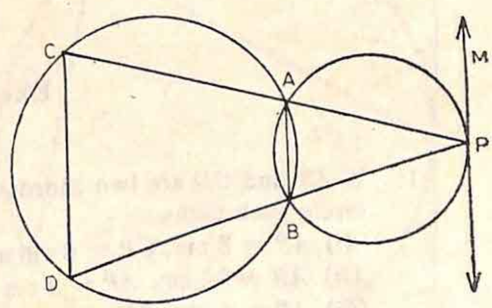



**Fig. 10.20**

*To prove :* $CD \parallel PM$

*Construction :* Join $A$ to $B$.

*Proof :* $\angle APM = \angle ABP$ (*i*)

(angle in alternate segment)

Also, $ABDC$ is a cyclic quadrilateral.

$\therefore$ $\angle ABD + \angle ACD = 180°$ (*ii*) (opposite angles of a cyclic quadrilateral)

But $\angle ABD + \angle ABP = 180°$ (*iii*) (linear pair)

From (*ii*) and (*iii*),

$\angle ABD + \angle ACD = \angle ABD + \angle ABP$

i.e. $\angle ACD = \angle ABP$ (*iv*)

$\therefore \qquad \angle ACD = \angle APM$ [From (*i*) and (*iv*)]

But these are alternate angles.

$\therefore \qquad CD \parallel PM$

**Example 5 :** Two circles intersect each other at the points $A$ and $B$. $CD$ is a direct common tangent line (Fig. 10.21) touching the circles at $C$ and $D$. Prove that the angles subtended by the segment $CD$ at $A$ and $B$ are supplementary.

**Solution :**

*Given* : Two circles intersecting at $A$ and $B$. $CD$ is a direct common tangent line (Fig. 10.21).

*To prove* : $\angle CAD + \angle CBD = 180°$

*Construction* : Join $A$ to $B$.

*Proof* : $\left.\begin{array}{l}\angle ACD = \angle CBA \\ \angle ADC = \angle ABD\end{array}\right\}$

(angles in the alternate segments)

$\therefore \quad \angle ACD + \angle ADC = \angle CBA + \angle ABD$

or $\quad \angle ACD + \angle ADC = \angle CBD \qquad ...(i)$

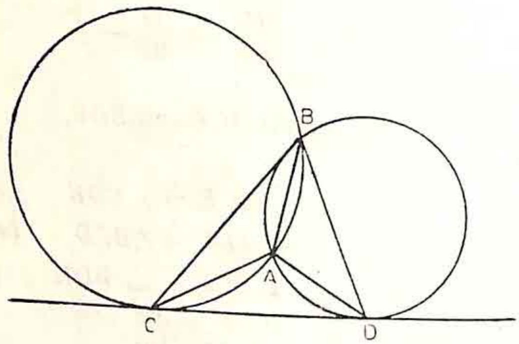



**Fig. 10.21**

Adding $\angle CAD$ to both sides of (*i*), we have

$$\angle ACD + \angle ADC + \angle CAD = \angle CBD + \angle CAD$$

But $\quad \angle ACD + \angle ADC + \angle CAD = 180°$ (sum of the angles of $\triangle ACD$)

Hence, $\quad \angle CAD + \angle CBD = 180°$

## Exercises 10.1

1. If $AB$ and $CD$ are two chords of a circle intersecting at a point $P$ inside the circle such that :
   (*i*) $AP = 8$ cm, $CP = 6$ cm and $PD = 4$ cm, find $PB$.
   (*ii*) $AB = 12$ cm, $AP = 2$ cm and $PD = 4$ cm, find $CP$.
   (*iii*) $AP = 6$ cm, $PB = 5$ cm and $CD = 13$ cm, find $CP$.
2. If $AB$ and $CD$ are two chords of a circle which when produced meet at a point $P$ such that :
   (*i*) $PA = 10$ cm, $PB = 4$ cm and $PC = 8$ cm, find $PD$.
   (*ii*) $PC = 15$ cm, $CD = 7$ cm and $PA = 12$ cm, find $AB$.
   (*iii*) $PA = 16$ cm, $PC = 10$ cm and $PD = 8$ cm, find $AB$.
3. If $AB$ and $CD$ are two chords intersecting at a point $P$ inside the circle such that $AP = CP$, show that $AB = CD$.

4. If $AB$ and $CD$ are two chords which when produced meet at a point $P$ and if $AP = CP$, show that $AB = CD$.

5. (*i*) In Fig. 10.22, $\angle ABC = 42°$. Find $\angle CAD$ and $\angle AEC$.

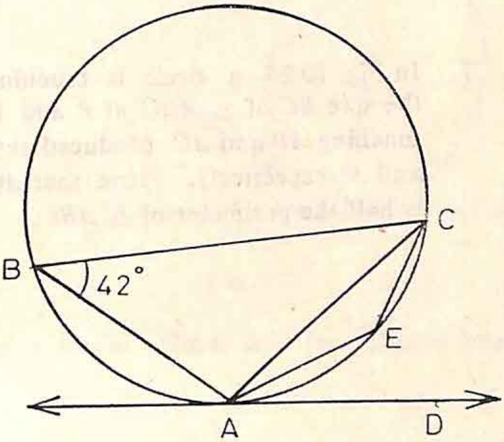



**Fig. 10.22**

(*ii*) In Fig. 10.23, $PT$ is the tangent line to a circle. If $\angle PTB = 60°$ and $\angle BTA = 45°$, find $\angle ABT$.

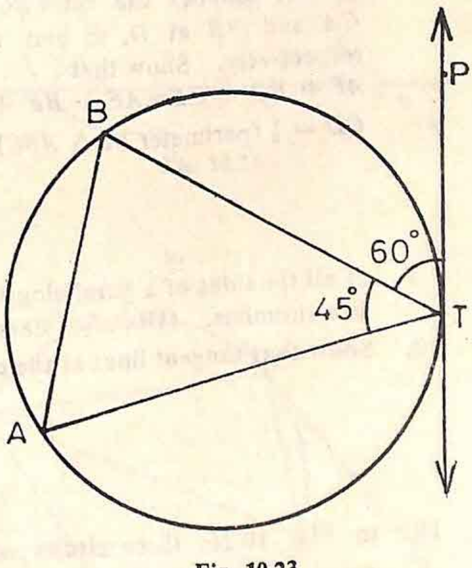



**Fig. 10.23**

6. Two circles touch each other externally. Prove that the lengths of the tangents drawn to the two circles from any point on the common tangent line at the point of contact of two circles are equal.

7. In Fig. 10.24, a circle is touching the side $BC$ of $\triangle ABC$ at $P$ and is touching $AB$ and $AC$ produced at $Q$ and $R$ respectively. Prove that $AQ$ is half the perimeter of $\triangle ABC$.

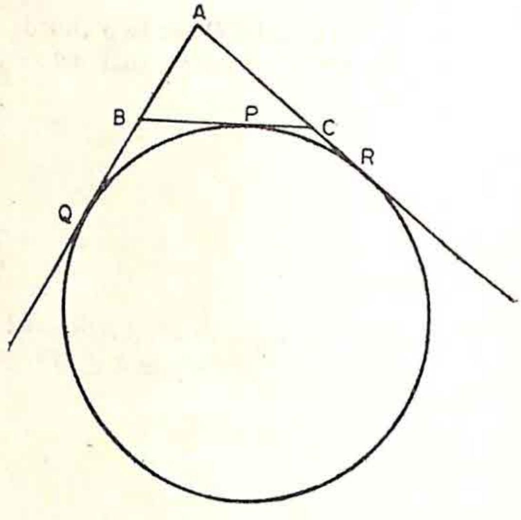



Fig. 10.24

8. In Fig. 10.25, the incircle of $\triangle ABC$, touches the sides $BC$, $CA$ and $AB$ at $D$, $E$ and $F$ respectively. Show that $AF + BD + CE = AE + BF + CD = \frac{1}{2}$ (perimeter of $\triangle ABC$).

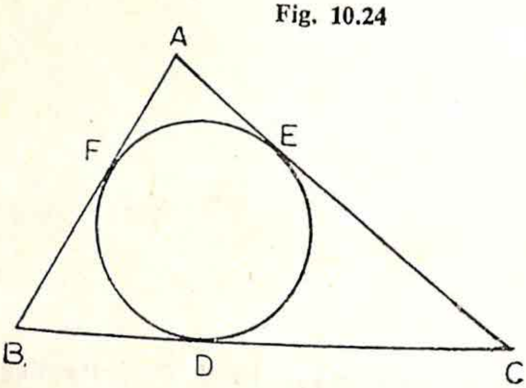
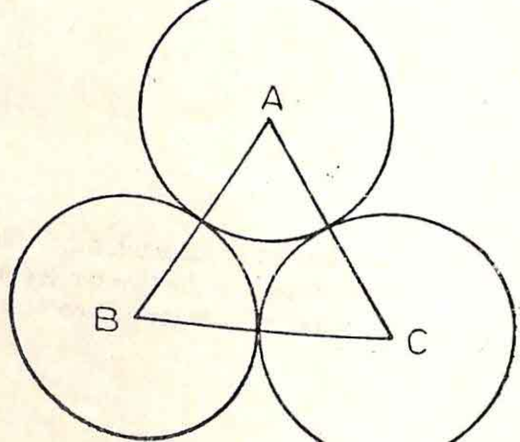



Fig. 10.25

9. If all the sides of a parallelogram touch a circle, show that the parallelogram is a rhombus. (**Hint**: See Example 1.)

10. Show that tangent lines at the end points of a diameter of a circle are parallel.

11. In Fig. 10.26, three circles with equal radii touch each other externally. Show that the triangle formed by joining their centres is an equilateral triangle.



12. In Fig. 10.27, $CD$ is the tangent line at $C$ to the circumcircle of $\Delta\ ABC$ intersecting $AB$ produced in $D$. Show that $\Delta DBC \sim \Delta DCA$.

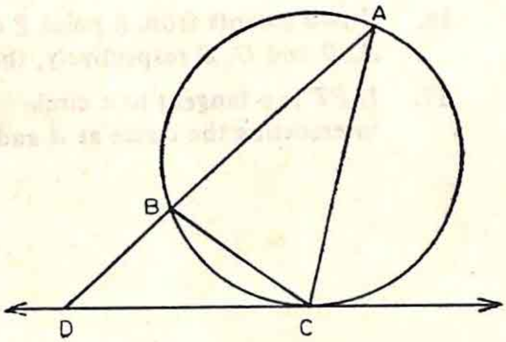



Fig. 10.27

13. $AB$ and $AC$ are two equal chords of a circle. Show that the tangent line at $A$ is parallel to $BC$.

14. In Fig. 10.28, $PT$ is a tangent line and $PAB$ is a secant to the circle. If the bisector of $\angle ATB$ intersects $AB$ at $M$, show that

(i) $\angle PMT = \angle PTM$
(ii) $PT = PM$

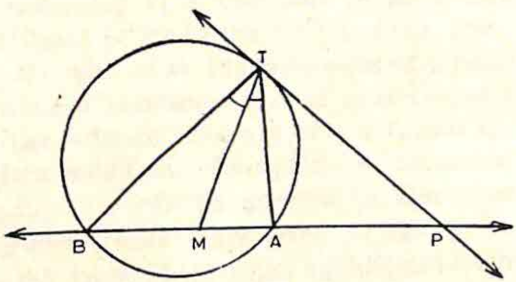



Fig. 10.28

15. In Fig. 10.29, $KLMN$ is a cyclic quadrilateral and $PQ$ is a tangent line to the circle at $K$. If $LN$ is a diameter of the circle, $\angle KLN = 30°$ and $\angle MNL = 60°$, find

(i) $\angle QKN$
(ii) $\angle LKN$
(iii) $\angle PKL$
(iv) $\angle LMN$
(v) $\angle MLN$

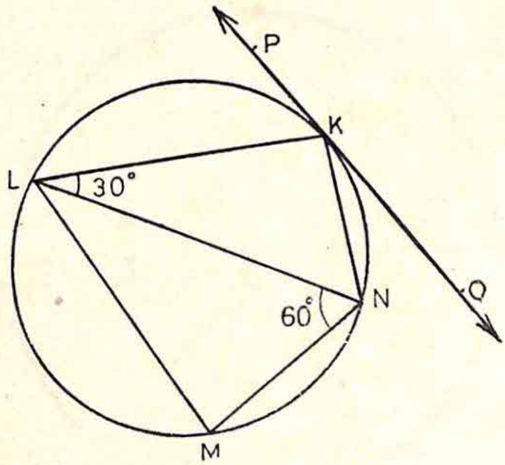



Fig. 10.29

16. If two secants from a point $P$ outside a circle intersect the circle at the points $A$, $B$ and $C$, $D$ respectively, then prove that $AP \times PB = PC \times DP$.

17. If $PT$ is a tangent to a circle from an exterior point $P$ and $PAB$ is a secant intersecting the circle at $A$ and $B$, then prove that $PT^2 = PA \times PB$.

CHAPTER 11

# Loci

## 11.1 Meaning of Locus

Loci is the plural of the word 'locus'. The meaning of the word locus, as understood in Mathematics, will be clear after going through the following examples :

1. Suppose the opposite banks of a river are regarded as straight lines. A boy walks along one of the banks, his foot always remaining at a constant perpendicular distance from the opposite bank, the constant distance '*d*' being the width of the river. The movement of the foot of the boy can be *thought of as the movement of a point* along a path such that its position always satisfies a condition viz. it is always at a constant distance '*d*' from the opposite bank. The different positions of the foot is the collection of points which always satisfies the given condition. This path or collection of points is termed as *Locus* of the moving point. It can be guessed in this case that the locus of the foot of the boy is a line parallel to the other bank of the river. An analogous situation is the set of points in the plane of the paper equidistant from a given line *l*. The locus of such points is a line parallel to the line *l*.

2. Recall that a circle of radius '*r*' is drawn on a paper with a compass, keeping its sharp steel end at a point *C* and the tip of the pencil starting from the point *A* (Fig. 11.1). The tip of the pencil *P* taking different positions on the circle, always remains at the same distance '*r*' from the point *C*. Thus, *P* can be regarded as a point which takes up different positions, starting from *A* and satisfying the condition that its position is at a constant distance '*r*' from the point *C*. This set of points, the positions of which always satisfy a given condition, is termed as the locus of the moving point *P*. The analogous situation is the movement of a bullock tied to a peg at a point *C* by a rope, moving with the rope taut. The locus of a foot (regarded as a point) of the bullock is a circle.

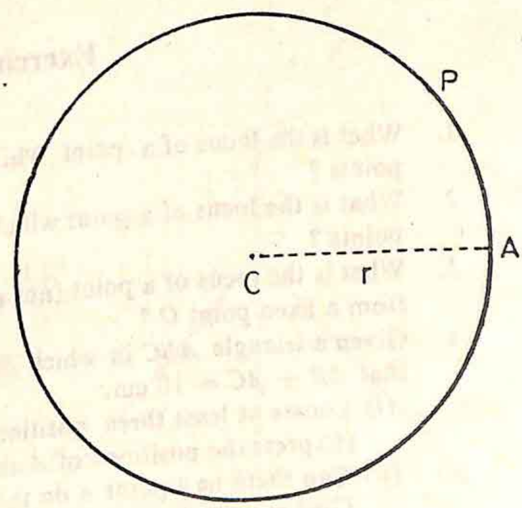



**Fig. 11.1**

The above examples suggest the following definition of the locus :

The locus of a point is the set of points whose positions satisfy a given condition.

Thus, in example 1, if $l_1$ is the line (one of the banks) and the foot $P$ of the boy moves such that it is always at a constant distance '$d$' from $l_1$, $Q$ being the foot of the perpendicular from $P$ to $l_1$, then locus of the moving foot $P$ is the line $l_2$ where

$l_2 = \{P \mid PQ = d, PQ \perp l_1, Q \in l_1\}$

Obviously, $l_2$ is a line parallel to $l_1$.

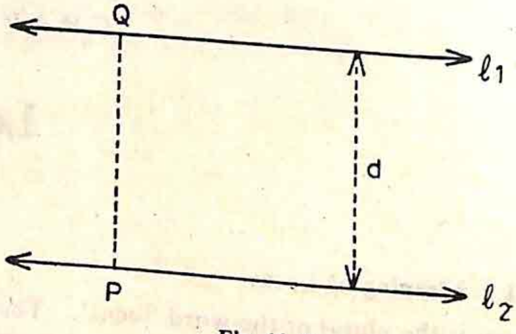



Fig. 11. 2

In example 2, the locus of the moving point $P$ is the set of points $S$ where $S = \{P \mid CP = r\}$. Obviously, $S$ is a circle.

In the chapter on circle in Class IX, it was seen that an infinite number of circles can be drawn passing through two given points $A$ and $B$. Let us see whether the collection of the centres of all such circles can be described as a locus. The centres for all such circles were seen to lie on a line which is the right bisector of the line segment $AB$. What conditions do these centres always satisfy ? It is clear that they are always equidistant from the points $A$ and $B$. Hence, in terms of locus, it can be said that the right bisector of the line segment $AB$ is the locus of the centre of a circle which passes through the points $A$ and $B$.

## Exercises 11.1

1. What is the locus of a point which is equidistant from three non-collinear points ?
2. What is the locus of a point which is equidistant from three distinct collinear points ?
3. What is the locus of a point (not necessarily in a plane) which is equidistant from a fixed point $O$ ?
4. Given a triangle $ABC$ in which $BC = 5$ cm, and $A$ is a moving point such that $AB + AC = 10$ cm.
   (*i*) Locate at least three positions of $A$ which satisfy the given condition. (Express the positions of $A$ as distances from the points $B$ and $C$.)
   (*ii*) Can there be a point $A$ on the locus such that $AB$=9 cm and $AC = 1$ cm ? Give reasons in support of your answer.

(*iii*) Suggest some more measurements of $AB$ and $AC$ such that $AB + AC = 10$ cm, but $A$ does not lie on the locus.

## 11.2 Theorems on Locus

We now study two theorems on locus. The theorems on locus are examples of characterisation theorems wherein it is essential to prove the theorem and its converse.

**Theorem : The locus of a point equidistant from two fixed points is the right bisector of the segment joining the points.**

*I. Given* : $A$ and $B$ are two distinct points and $P$ is a point such that

$AP = BP$ (Fig. 11.3).

*To prove* : $P$ lies on the right bisector of $AB$.

*Construction* : Join $P$ to the mid-point $C$ of $AB$. Join $P$ to $A$ and $P$ to $B$.

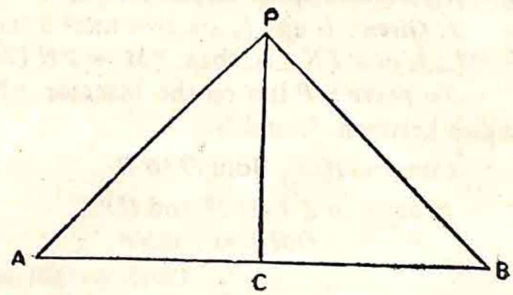


Fig. 11.3

*Proof* : In triangles $APC$ and $BPC$,

$AP = BP$ (given)
$AC = BC$ ($C$ is mid-point of $AB$)
$PC = PC$
$\therefore \triangle APC \cong \triangle BPC$ ($SSS$ criterion)
$\therefore \angle ACP = \angle BCP$ (corresponding angles)

But these angles form a linear pair.

Hence, $\angle ACP = \angle BCP = 1$ right angle

$\therefore$ $CP$ is the right bisector of $AB$.

Hence, $P$ lies on the right bisector of $AB$.

*Conversely*

*II. Given* : $A$ and $B$ are two distinct points and $P$ is any point on the right bisector $CR$ of $AB$ (Fig. 11.4).

*To prove* : $AP = BP$

*Construction* : Join $P$ to $A$ and also $P$ to $B$.

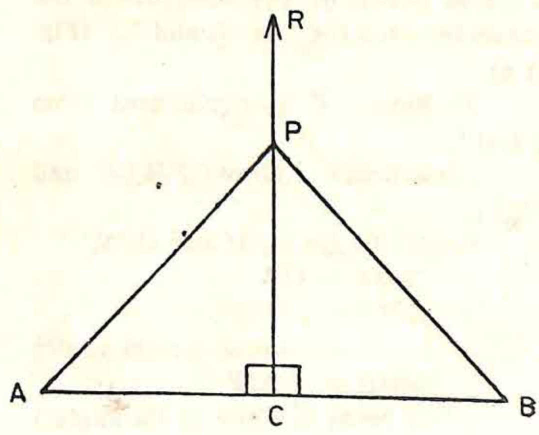


Fig. 11.4

*Proof* : In $\triangle$s $APC$ and $BPC$,

$AC = BC$ ($C$ is mid-point of $AB$)
$PC = PC$
$\angle ACP = \angle BCP$ (each a right angle)
$\therefore \triangle APC \cong \triangle BPC$
($SAS$ criterion)

Hence, $AP = BP$

*i.e.* $P$ is equidistant from $A$ and $B$
or $P$ lies on the locus.

The above theorem proves that the locus of a point equidistant from two fixed points is the right bisector of the line segment joining the fixed points and conversely any point on the right bisector is equidistant from the fixed points.

*Note* : Having proved the above theorem, it becomes simple to see that infinite number of circles can be drawn passing through two given points. Centres of such circles lie on the right bisector of the line segment joining the points.

**Theorem : The locus of a point equidistant from two intersecting lines is the pair of bisectors of the angles formed by the given lines.**

*I. Given* : $l_1$ and $l_2$ are two lines intersecting at $O$ and $P$ is a point on the locus *i.e.* if $PM \perp l_1$ and $PN \perp l_2$, then $PM = PN$ (Fig. 11.5).

*To prove* : $P$ lies on the bisector of the angles between $l_1$ and $l_2$.

*Construction* : Join $O$ to $P$.

*Proof*: In $\triangle s$ $OPM$ and $OPN$,

$\angle OMP = \angle ONP$ (each a right angle)

$PM = PN$ (given)

$OP = OP$

$\therefore \quad \triangle OPM \cong \triangle OPN$ ($RHS$ criterion)

$\therefore \quad \angle MOP = \angle NOP$ (corresponding angles)

Hence, $P$ is on the bisector of $\angle MON$.

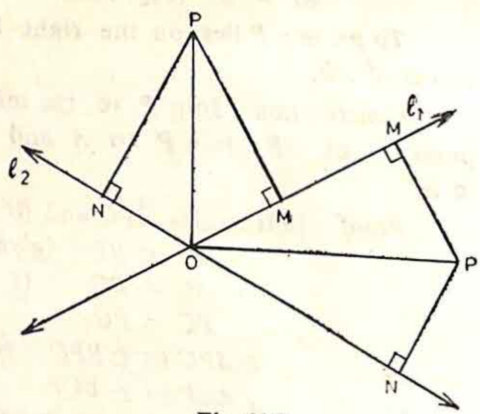


**Fig. 11.5**

*Conversely*

*II. Given* : $l_1$ and $l_2$ are two lines intersecting at a point $O$ and $P$ is any point on either of the bisectors of the angles between the lines $l_1$ and $l_2$. (Fig. 11.6).

*To prove* : $P$ is equidistant from $l_1$ and $l_2$.

*Construction* : Draw $PM \perp l_1$ and $PN \perp l_2$.

*Proof* : In $\triangle s$ $OPM$ and $OPN$,

$OP = OP$

$\angle OMP = \angle ONP$ (each a right angle)

$\angle MOP = \angle NOP$ ($OP$ being bisector of the angles)

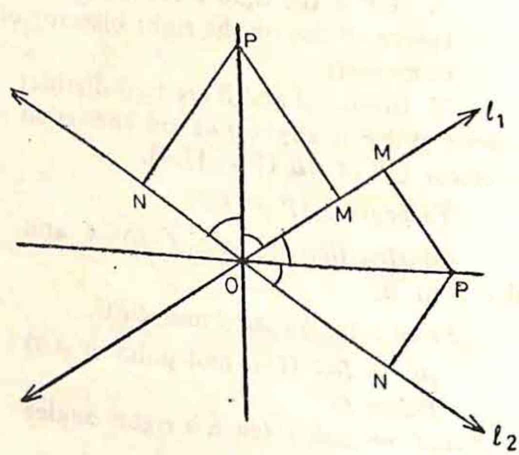


**Fig. 11.6**

$\therefore \qquad \triangle OPM \cong \triangle OPN$ (*ASA* criterion)

Therefore, $\qquad PM = PN$ (corresponding sides)

*i.e.* $P$ is equidistant from the lines $l_1$ and $l_2$.

Hence, any point on the bisectors of the angles is equidistant from the lines $l_1$ and $l_2$.

The above theorem proves that locus of a point equidistant from two intersecting lines is the pair of bisectors of the angles between the lines.

As an application of the above theorem, we prove the following result. (This result has already been proved in chapter 9 as an exercise using similarity of triangles.)

**Theorem : The bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle.**

*Given* : A $\triangle ABC$ in which $AD$ is the bisector of $\angle BAC$ (Fig. 11.7).

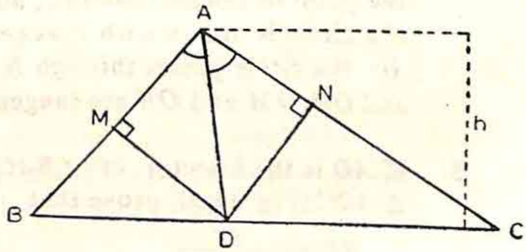


**Fig. 11.7**

*To prove* : $\dfrac{BD}{CD} = \dfrac{AB}{AC}$

*Construction* : Draw $DM \perp AB$ and $DN \perp AC$

*Proof*: Since $\triangle ABD$ and $\triangle ACD$ have same altitudes say $h$,

$$\frac{\text{Area }(\triangle ABD)}{\text{Area }(\triangle ACD)} = \frac{\frac{1}{2}\, BD \times h}{\frac{1}{2}\, CD \times h} = \frac{BD}{CD} \qquad \ldots(i)$$

Also, since $D$ lies on the bisector of $\angle BAC$,

$\therefore \qquad DM = DN$

$$\text{Now, } \frac{\text{Area }(\triangle ABD)}{\text{Area }(\triangle ACD)} = \frac{\frac{1}{2}\, AB \times DM}{\frac{1}{2}\, AC \times DN} = \frac{AB}{AC} \qquad \ldots(ii)$$

Hence, from (*i*) and (*ii*),

$$\frac{BD}{CD} = \frac{AB}{AC}$$

In case, $AD$ is the bisector of exterior $\angle A$ *i.e.* of $\angle LAC$ (Fig. 11.8), then $D$ divides $BC$ externally in the ratio $AB : AC$. (The proof is exactly the same as above.)

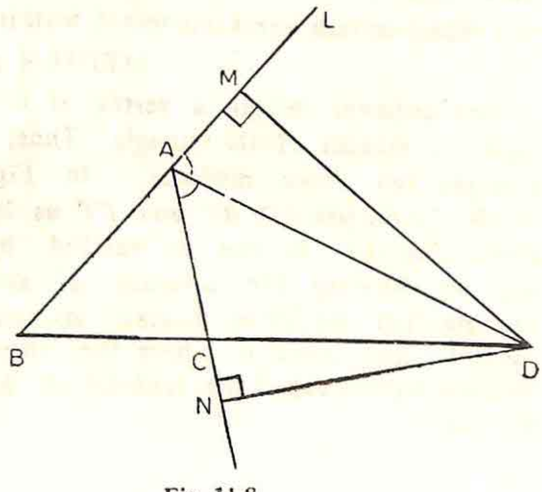


Fig. 11.8

## Exercises 11.2

1. What is the locus of a point equidistant from two lines which are parallel ?
2. Find the locus of the vertex $A$ of an isosceles triangle $ABC$ which has $BC$ as its fixed base. Can we conclude that the perpendicular from the vertex $A$ of an isosceles triangle $ABC$ bisects the base $BC$ ?
3. What is the condition that a circle passes through four given non-collinear points ? Find a similar condition for the circle to pass through the vertices of a $n$-sided polygon.
4. $P$ is any point on a bisector of the angles between the lines $l_1$ and $l_2$ and $O$ is the point of intersection of $l_1$ and $l_2$. If $PM \perp l_1$ and $PN \perp l_2$ (Fig. 11.5) and if a circle is drawn with $P$ as centre and $PM$ as radius, prove that
   (*i*) the circle passes through $N$
   and (*ii*) $OM$ and $ON$ are tangents to the circle.
5. If $AD$ is the bisector of $\angle BAC$ of $\triangle ABC$ (Fig. 11.9), prove that
$$BD = \frac{ac}{b+c}$$
$$CD = \frac{ab}{b+c}$$

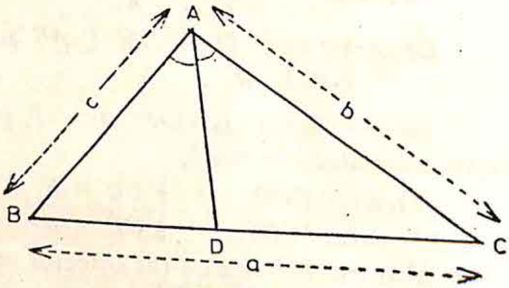



Fig. 11.9

6. In $\triangle ABC$, $D$ is a point on $BC$ such that
$$\frac{BD}{DC} = \frac{AB}{AC}.$$
Prove that $AD$ bisects $\angle BAC$.

### 11.3 Some special Points connected with a Triangle

We define certain terms connected with triangles.

MEDIAN AND CENTROID

A line segment joining a vertex of a triangle to the mid-point of the opposite side is called a **median** of the triangle. Thus, a triangle has three medians. In Fig. 11.10, $\triangle ABC$ has $AD$, $BE$ and $CF$ as its three medians. It can be verified by actually drawing the medians of any triangle that the three medians are concurrent. The point $G$ where the three medians meet is called the **centroid** of the triangle.

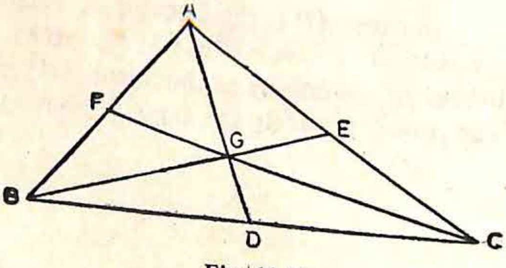



Fig. 11.10

## ORTHOCENTRE

The perpendicular drawn from a vertex of a triangle to the opposite side is called the **altitude** of the triangle from that vertex. Thus, a triangle has three altitudes. In Fig. 11.11, *AD*, *BE* and *CF* are the three altitudes of triangle *ABC*. It can again be verified by actual construction that the three altitudes meet at a point. The point of concurrence of three altitudes is called the **orthocentre** of the triangle.

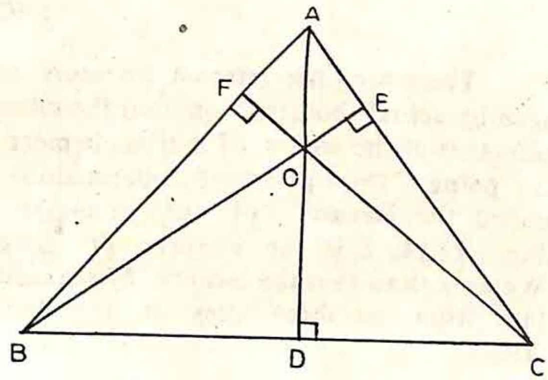



Fig. 11.11

It may be noted that the orthocentre *O* of a triangle may lie inside the triangle as in Fig. 11.11, or outside the triangle as in Fig. 11.12 or on the triangle as in Fig. 11.13 where the triangle is a right triangle.

In Fig. 11.13, the vertex *B* is the orthocentre of the triangle. Can the centroid of a triangle lie on the triangle or outside the triangle ? Certainly not. It always lies

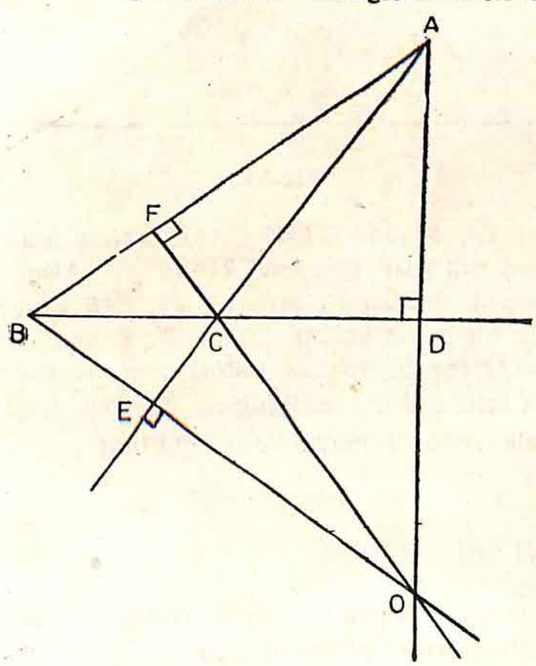



Fig. 11.12

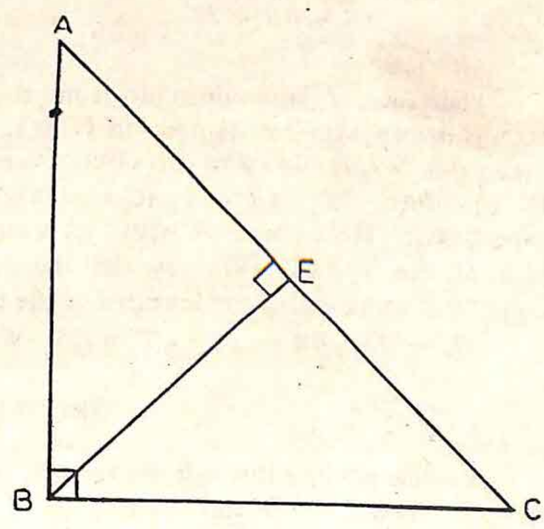



Fig. 11.13

inside the triangle. Can the orthocentre lie at a point on the sides of a triangle other than the vertices ?

## INCENTRE

There are three internal bisectors of the angles of a triangle. Again it can be seen by actual construction that the internal bisectors of the angles of a triangle meet at a point. This point of intersection is called the **incentre** of the triangle. In Fig. 11.14, $I$ is the incentre of $\triangle ABC$. We now show that the incentre $I$ is equidistant from the three sides of the triangle $ABC$.

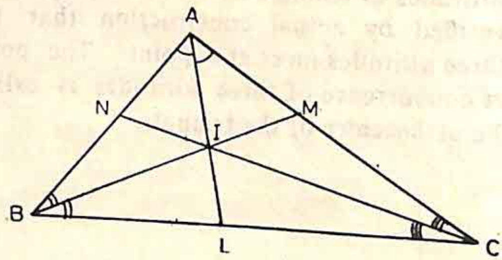



**Fig. 11.14**

*Proof* : Consider the incentre $I$ of the triangle $ABC$. Draw $IR \perp AB$, $IS \perp AC$ and $IT \perp BC$ (Fig. 11.15).

$I$ lies on the bisector of $\angle BAC$.

Therefore, $I$ is equidistant from $AB$ and $AC$.

$\therefore \quad IR = IS \quad \ldots(i)$

Similarly, $I$ lies on bisector of $\angle ABC$.

$\therefore \quad IR = IT \quad \ldots(ii)$

$\therefore$ From $(i)$ and $(ii)$,

$$IR = IS = IT$$

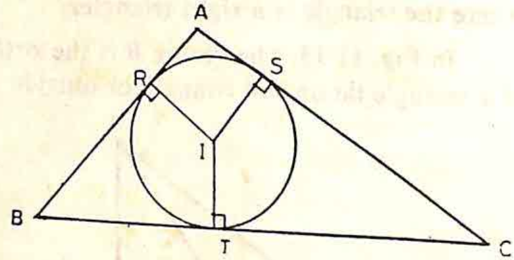



**Fig. 11.15**

Therefore, $I$ is equidistant from the sides $AB$, $BC$ and $CA$ of $\triangle ABC$. Now if a circle is drawn with $I$ as centre and $IT$ as radius, it will pass through $R$ and $S$. Also, since radii $IT$, $IR$ and $IS$ of this circle are respectively perpendiculars to $BC$, $AB$ and $AC$, therefore, $BC$, $AB$ and $AC$ are tangents to the circle at the points $T$, $R$ and $S$ respectively. Hence, a circle with $I$ as centre and $IT$ (or $IR$, $IS$) as radius touches the sides of the $\triangle ABC$. We say that the circle is **inscribed** in the triangle. This justifies the point $I$ being called the incentre of the triangle $ABC$. It may be observed that

$AR = AS$ ; $BR = BT$ ; $CT = CS$. (Why ?)

## CIRCUMCENTRE

A circle passing through the vertices of a triangle is called the **circumcircle** of the triangle. The centre of this circle is called the **circumcentre** of the triangle.

It can again be seen by actual construction that the right bisectors of the sides of any triangle $ABC$ (as in Fig. 11.16) meet at a point $S$. We now show that this point of concurrence $S$ is the circumcentre of $\triangle ABC$.

In other words, we prove that the point of intersection of the right bisectors of the sides of a triangle $ABC$ is the circumcentre $S$ of triangle $ABC$.

*Proof*: Since $S$ lies on the right bisector of $BC$, $S$ is equidistant from $B$ and $C$.

$\therefore \quad BS = CS \quad ...(i)$

Also, $S$ lies on the right bisector of $AB$.

$\therefore \quad AS = BS \quad ...(ii)$

From $(i)$ and $(ii)$,

$$AS = BS = CS$$

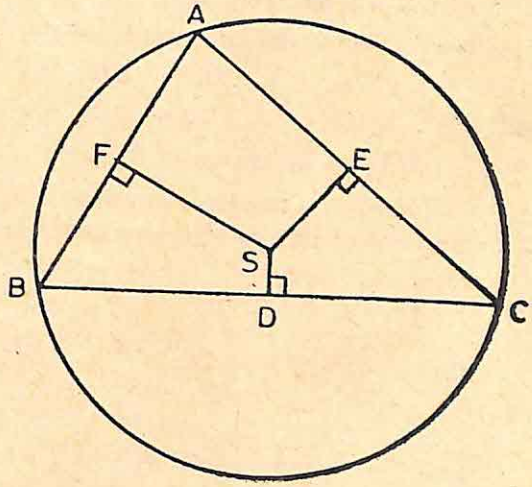



Fig. 11.16

Hence, $S$ is equidistant from the vertices of the $\triangle ABC$.

Hence, $S$ is the circumcentre of the $\triangle ABC$.

Now let us prove one theorem regarding the medians of a triangle.

**Theorem: The medians of a triangle pass through the same point which divides each of the medians in the ratio 2 : 1.**

*Given*: A $\triangle ABC$ in which medians $BE$ and $CF$ intersect at $G$ (Fig. 11.17).

*To prove*: $G$ lies on the third median $AD$ and $G$ divides each of these medians in the ratio 2 : 1,

*i.e.* $$\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = \frac{2}{1}$$

*Construction*: Join $A$ to $G$ and produce it to $H$ such that $AG = GH$. Join $B$ to $H$ and also $C$ to $H$. Let $AH$ intersect $BC$ at $D$.

*Proof*: In $\triangle ABH$, $F$ is the mid-point of $AB$ and $G$ is the mid-point of $AH$ (by construction)

$\therefore \quad FG \parallel BH$

$\therefore \quad GC \parallel BH \quad ...(i)$

Similarly in $\triangle ACH$, $E$ is the mid-point of $AC$ and $G$ is the mid-point of $AH$.

$\therefore \quad EG \parallel HC$

$\therefore \quad BG \parallel HC \quad ...(ii)$

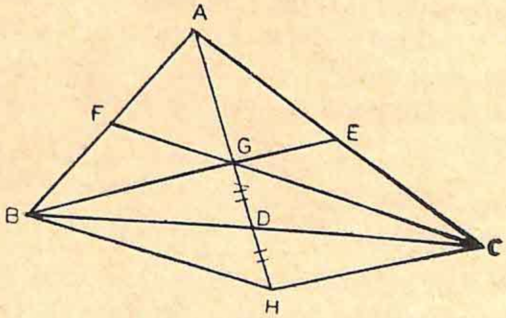



Fig. 11.17

From (*i*) and (*ii*), we get

*BGCH* is a parallelogram.

Now $D$ being the intersection of the diagonals of parallelogram *BGCH*, we have

$$GD = DH$$

and $$BD = DC$$

$\therefore$ $D$ is the mid-point of $BC$.

$\therefore$ $AD$ is the third median of $\triangle ABC$ which also passes through $G$. Hence, the three medians of the triangle are concurrent, *i.e.* pass through the same point.

Now, $GD = DH$

Also, $AG = GH$ (by construction)

$\therefore$ $AG = 2\,GD$

$\therefore$ $\dfrac{AG}{GD} = \dfrac{2}{1}$

*i.e.* $AG : GD = 2 : 1$

$\therefore$ The centroid $G$ divides the median $AD$ in the ratio 2 : 1. Similarly, it can be proved that $G$ divides each of the other two medians in the ratio 2 : 1.

## SOLVED EXAMPLE

**Example :** $O$ is a point on the right bisector of $AB$ such that $\angle AOB = 2\theta$ $(2\theta < \pi)$. Prove that the locus of a point $P$ such that $\angle APB = \theta$ is the major arc *ACPB* of a circle with centre $O$ and radius $OA$ (Fig. 11.18).

*Or*

If $O$ is a point on the right bisector of $AB$ such that $\angle AOB = 2\theta$ $(2\theta < \pi)$ and $P$ is a point such that $\angle APB = \theta$, then prove that $O$ is the circumcentre of $\triangle ABP$.

*Given* : $O$ is a point on the right bisector of $AB$ such that $\angle AOB = 2\theta$. $P$ is any point such that $\angle APB = \theta$.

*To prove* : Locus of $P$ is the major arc *ACPB* of a circle with centre $O$ and radius $OA$.

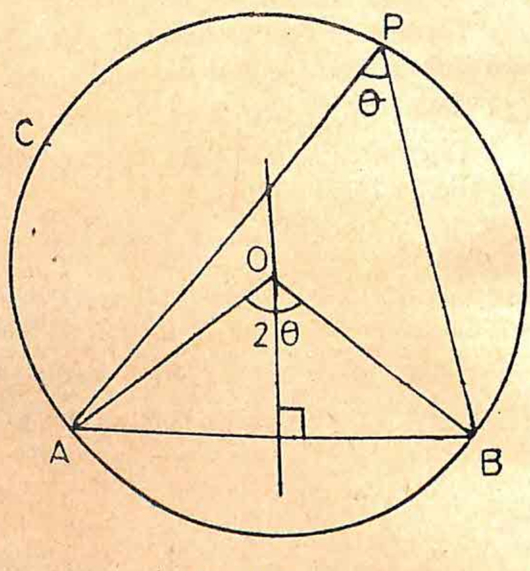



**Fig. 11.18**

*Construction* : With $O$ as centre and $OA$ $(= OB)$ as radius draw a circle. If it does not pass through $P$, let it cut $AP$ (Fig. 11.19) or $AP$ produced (Fig. 11.20) in the point $Q$. Join $B$ to $Q$.

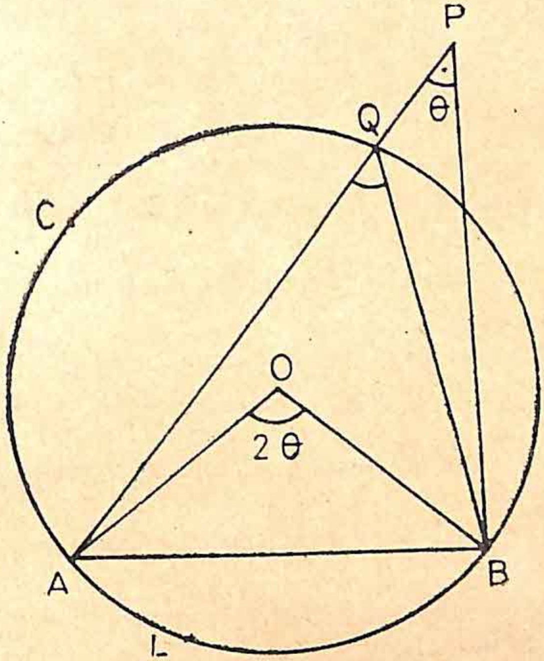



Fig. 11.19

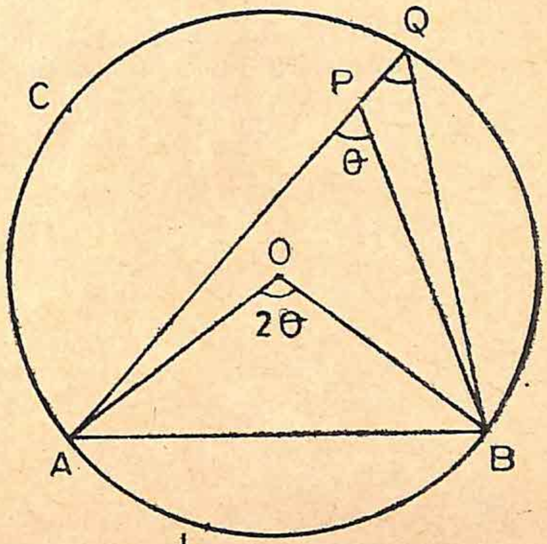



Fig. 11.20

*Proof*: If the circle passes through $P$ as in Fig. 11.18, what is required is proved. Suppose the circle does not pass through $P$ as in Figs. 11.19 and 11.20. Then,

$$\angle AQB = \frac{1}{2}\angle AOB = \theta$$

(angle at the centre is double the angle at a point on the remaining part of the circle.)

But $\quad \angle APB = \theta \quad$ (given)

$\therefore \quad \angle APB = \angle AQB$

which is impossible as exterior angle of a triangle is always greater than interior opposite angles.

Hence, the circle with $O$ as centre and $OA$ as radius must pass through $P$ or $P$ lies on the major arc $ACPB$ of the circle with $O$ as centre and $OA$ as radius. Hence locus of $P$ is the major arc $ACPB$ of the circle with centre $O$ and radius $OA$.

*Note*: It can be proved similarly that locus of a point $P$ such that $\angle APB = 180° - \theta$, is the arc $ALB$ of the circle with centre $O$ and radius $OA$.

## Exercises 11.3

1. $A$ and $B$ are two distinct points. Prove that the locus of a point $P$ such that $\angle APB = 90°$ is a circle with mid-point $O$ of $AB$ as centre and $OA$ as radius.
2. Prove that the locus of the vertex of a triangle of given base $AB$ and height $h$ is a pair of lines parallel to the base.
3. Prove that the locus of the vertex of a triangle whose base is $AB$ and area $Z$ is a pair of lines parallel to the base.
4. $ABC$ is a triangle of area $Z$ and $r$ is the radius of its incircle. Prove that $r = \frac{Z}{S}$ where $2S = AB + BC + CA$.
5. Find the locus of the centre of a circle of radius $r$ touching externally a circle of radius $R$.
6. Find the locus of the centre of a circle of radius $r$ touching internally a circle of radius $R$.
7. Prove that the locus of the mid-points of the chords of a circle equidistant from the centre is a circle of radius equal to the distance of the chords from the centre of the given circle.

CHAPTER 12

# Constructions

## 12.1 Introduction

In chapter 10 we studied many geometrical results. We shall utilise these results for the construction of tangent lines to a circle. As before, in all constructions only ruler and compasses shall be used as instruments. In every construction you are expected to write down the essential steps of constructions.

## 12.2 Construction of Tangent Lines

**Construction 1 :** To draw a tangent line to a circle at a given point on the circle.

*Given* : A circle with centre $O$ and a point $P$ on it (Fig. 12.1).

*Required* : To construct the tangent line to the circle at $P$.

*Steps of Construction* :

1. Join $O$ to $P$.
2. At $P$, draw $PT \perp OP$.

Then, $PT$ is the required tangent line to the circle at the point $P$.

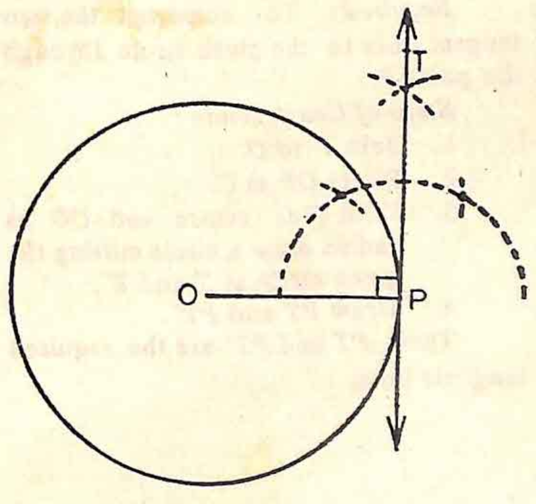



**Fig. 12.1**

**Construction 2 :** To draw the tangent line from a point on a circle whose centre is not marked.

*Given* : A circle and a point $P$ on it (Fig. 12.2).

*Required*: To construct the tangent line at $P$.

*Steps of Construction* :

1. Through $P$, draw a chord $PR$.
2. Make any angle say $\angle PQR$ in the major segment.
3. At $P$, draw $\angle RPT = \angle PQR$ such that $T$ and $Q$ lie on opposite sides of $PR$.

Then, $PT$ is the required tangent line.

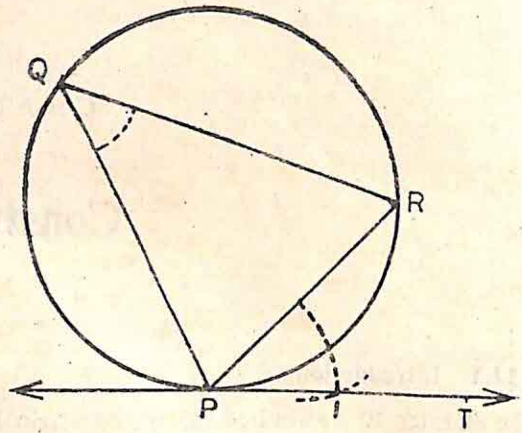



Fig. 12.2

**Construction 3 :** To draw tangent lines to a circle through a point outside the circle.

*Given* : A circle with centre $O$ and a point $P$ outside the circle (Fig. 12.3).

*Required* : To construct the two tangent lines to the given circle through the point $P$.

*Steps of Construction* :

1. Join $P$ to $O$.
2. Bisect $OP$ at $C$.
3. With $C$ as centre and $CO$ as radius draw a circle cutting the given circle at $T$ and $T'$.
4. Draw $PT$ and $PT'$.

Then, $PT$ and $PT'$ are the required tangents lines.

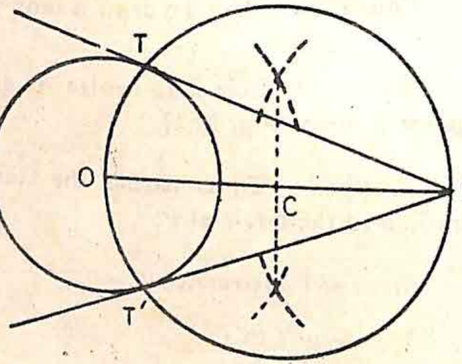



Fig. 12.3

## Exercises 12.1

1. Draw a tangent to a circle of diameter 5 cm at a point $P$ on it.
2. Draw two tangents to a circle of radius 4 cm from a point $P$ at a distance of 6 cm from its centre. Also measure the lengths of the two tangents. Are they equal ? Give reasons for your answer.

3. Draw two tangents at the extremities of diameter $AB$ of a circle of radius 1.5 cm. Are these tangents parallel ? Give reasons for your answer.
4. Draw two tangents to a circle whose diameter is 6 cm from a point $P$ at a distance of 9 cm from its centre.
5. Draw a tangent at a point $P$ of a circle of radius 3 cm without using the centre.
6. Construct the incircle of $\triangle ABC$ in which $BC = 7$ cm, $CA = 5$ cm and $AB = 6$ cm.
7. Construct the circumcircle of $\triangle ABC$ in which $BC = 7$ cm, $CA = 5$ cm and $AB = 6$ cm.
8. Construct a triangle with base $BC=6$ cm and vertex $A$ such that $\angle BAC=60°$. How many such triangles can be constructed ?

CHAPTER 13

# Trigonometrical Ratios

### 13.1 Introduction

The word 'Trigonometry' is derived from two Greek words 'trigon' and 'metron'. Here 'tri' means three, 'gon' means sides and 'metron' means 'a measure'. Thus, trigonometry is that branch of Mathematics which deals with the measurement of the sides and angles of a triangle.

Knowledge of trigonometry is of great help in surveying, geography, astronomy, physics, navigation, etc. The captains of ships work out the distances of sea-shores and that of other ships sailing in the oceans with the help of trigonometry.

### 13.2 Angle

In geometry we consider, generally, angles which are less than 180° and on a few occasions angles greater than 180° but not greater than 360°. However, in practical situations concerning rotating or turning bodies such as wheels, pivoted needles etc., we need angles of any magnitude (Fig. 13.1).

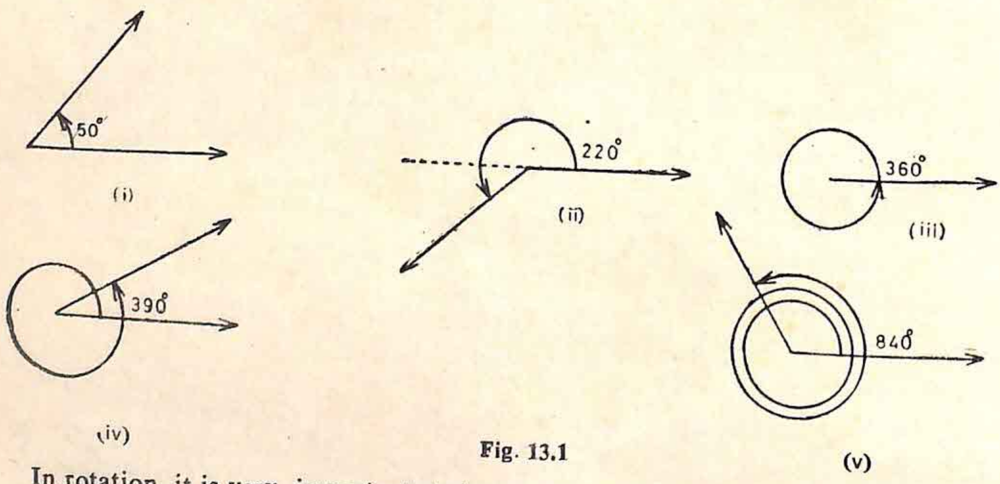



Fig. 13.1

In rotation, it is very important to know about the starting position and the final position of the ray, and also whether the rotation is in the 'clockwise' direction or 'anti-clockwise' direction.

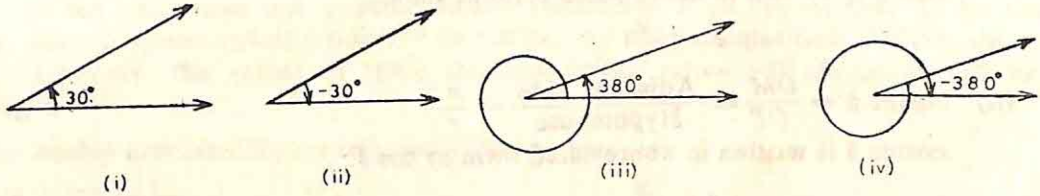



**Fig. 13.2**

In figure 13.2 (*i*), the angle is formed by 'anticlockwise' rotation. Angles formed by anticlockwise rotations are taken to be positive angles. So the measure of the angle in figure 13.2 (*i*) is taken as $+30°$ (or simply $30°$). In the same way, the measure of the angle in figure 13.2 (*iii*) is $380°$. Now in figure 13.2 (*ii*) the angle is formed by clockwise rotation. Angles formed by clockwise rotations are taken to be negative angles. So the measure of the angle in figure 13.2 (*ii*) is $-30°$ and the measure of the angle in figure 13.2 (*iv*) is $-380°$.

In this chapter, we will deal with positive acute angles only. Recall that angles are measured either in degrees or in right angles.

A right angle is divided into 90 equal parts and each part is called a degree. A 'degree' is further divided into 60 equal parts and each part is known as a 'minute'. A 'minute' is further divided into 60 equal parts and each part is known as a 'second'. Thus,

**1 right angle = 90 degrees (90°)**
**1 degree = 60 minutes (60′)**
**1 minute = 60 seconds (60″)**

In this chapter, degree is taken as the unit of measure of an angle.

## 13.3 Trigonometrical Ratios of Angles

In a rectangular coordinate system $XOY$ (Fig. 13.3), consider a ray $OA$ and let $P(x, y)$ be a point on it. Draw $PM \perp OX$.

$\therefore\ OM = x$ and $PM = y$.

Let $\angle MOP = \theta$ and $OP = r$.

In right triangle $OMP$, $OM$ is called the adjacent side and $MP$ is called the opposite side with respect to the angle $\theta$. $OP$ is the hypotenuse of the triangle.

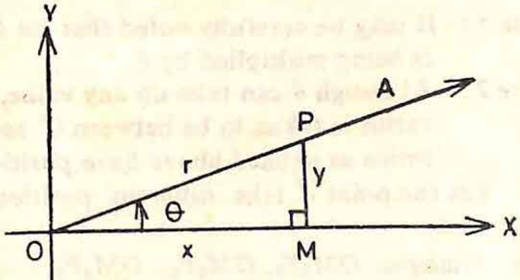



Fig. 13.3

Making use of lengths of the sides of the right triangle $OPM$, we define the following trigonometrical ratios of angle $\theta$.

(*i*) sine $\theta = \dfrac{MP}{OP} = \dfrac{\text{Opposite side}}{\text{Hypotenuse}} = \dfrac{y}{r}$

sine $\theta$ is written in abbreviated form as sin $\theta$.

$$\therefore \qquad \sin\theta = \frac{y}{r}$$

(*ii*) $\text{cosine } \theta = \frac{OM}{OP} = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{x}{r}$

cosine $\theta$ is written in abbreviated form as cos $\theta$.

$$\therefore \qquad \cos\theta = \frac{x}{r}$$

(*iii*) $\text{tangent } \theta = \frac{MP}{OM} = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{y}{x}$

tangent $\theta$ is written in abbreviated form as tan $\theta$.

$$\therefore \qquad \tan\theta = \frac{y}{x}$$

(*iv*) $\text{cotangent } \theta = \frac{OM}{MP} = \frac{\text{Adjacent side}}{\text{Opposite side}} = \frac{x}{y}$

cotangent $\theta$ is written in abbreviated form as cot $\theta$.

$$\therefore \qquad \cot\theta = \frac{x}{y}$$

(*v*) $\text{secant } \theta = \frac{OP}{OM} = \frac{\text{Hypotenuse}}{\text{Adjacent side}} = \frac{r}{x}$

secant $\theta$ is written in abbreviated form as sec $\theta$.

$$\therefore \qquad \sec\theta = \frac{r}{x}$$

(*vi*) $\text{cosecant } \theta = \frac{OP}{MP} = \frac{\text{Hypotenuse}}{\text{Opposite side}} = \frac{r}{y}$

cosecant $\theta$ is written in abbreviated form as cosec $\theta$.

$$\therefore \qquad \operatorname{cosec}\theta = \frac{r}{y}$$

*Note 1*: It may be carefully noted that sin $\theta$, cos $\theta$, etc. do not mean that sin or cos is being multiplied by $\theta$.

*Note 2*: Although $\theta$ can take up any value, in this chapter its value for trigonometrical ratios is taken to be between 0° and 90°. For such angles, the trigonometrical ratios as defined above have positive values.

Let the point $P$ take different positions on the ray $OA$, say, $P_1$, $P_2$, $P_3$, etc. (Fig. 13.4).

The triangles $OM_1P_1$, $OM_2P_2$, $OM_3P_3$, etc. are all similar. (Why) ?

Thus, $\frac{M_1P_1}{OP_1} = \frac{M_2P_2}{OP_2} = \frac{M_3P_3}{OP_3} = \ldots$

Hence, sin $\theta$ is the same irrespective of whether it is from $\Delta OM_1P_1$ or $\Delta OM_2P_2$ or $\Delta OM_3P_3$, etc.

Similarly, $\frac{OM_1}{OP_1} = \frac{OM_2}{OP_2} = \frac{OM_3}{OP_3} = \ldots$

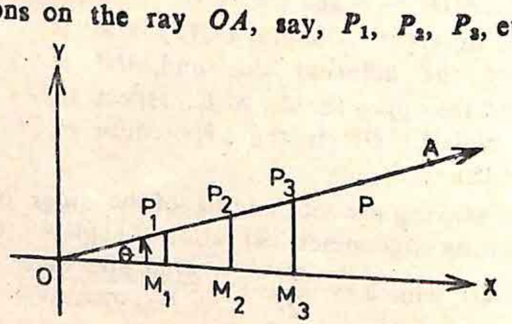



Fig. 13.4

So $\cos\theta$ also does not depend on the position of $P$ on the ray $OA$. In the same way, other trigonometrical ratios of $\theta$ do not depend upon the position of $P$ on the ray $OA$. However, the values of these trigonometrical ratios will change if angle itself changes.

**Relationship between different trigonometrical ratios**

From figure 13.3,

(i) $$\tan\theta = \frac{MP}{OM} = \frac{\frac{MP}{OP}}{\frac{OM}{OP}} = \frac{\sin\theta}{\cos\theta}$$

(ii) $$\cot\theta = \frac{OM}{MP} = \frac{\frac{OM}{OP}}{\frac{MP}{OP}} = \frac{\cos\theta}{\sin\theta}$$

(iii) $$\sec\theta = \frac{OP}{OM} = \frac{1}{\frac{OM}{OP}} = \frac{1}{\cos\theta}$$

(iv) $$\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{1}{\frac{MP}{OP}} = \frac{1}{\sin\theta}$$

(v) $$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\frac{\sin\theta}{\cos\theta}} = \frac{1}{\tan\theta}$$

(vi) $$\sin\theta = \frac{1}{\operatorname{cosec}\theta}$$

(vii) $$\cos\theta = \frac{1}{\sec\theta}$$

(*viii*) $\tan\theta = \dfrac{1}{\cot\theta}$

**13.4 Fundamental Identities**

From the right triangle $OMP$ in figure 13.5, we have :

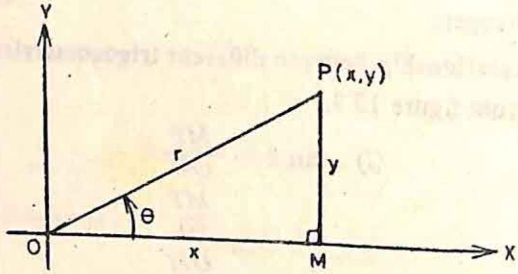


Fig. 13.5

$$x^2 + y^2 = r^2 \text{ (Why ?)} \quad \ldots(i)$$

(*i*) Dividing both sides of (*i*) by $r^2$, we have :

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

*i.e.* $\left(\dfrac{x}{r}\right)^2 + \left(\dfrac{y}{r}\right)^2 = 1$

*i.e.* $\mathbf{\cos^2\theta + \sin^2\theta = 1}$

*Note* : $\dfrac{x}{r} = \cos\theta$. Therefore, $\left(\dfrac{x}{r}\right)^2 = (\cos\theta)^2$.

But we have written

$$\left(\frac{x}{r}\right)^2 = \cos^2\theta. \quad \text{Note that } (\cos\theta)^2 \text{ is written as } \cos^2\theta.$$

In other words, $\cos^2\theta$ means $(\cos\theta)^2$. In the same way $\cos^3\theta$ means $(\cos\theta)^3$, $\sin^2\theta$ means $(\sin\theta)^2$ and so on.

(*ii*) Dividing both sides of (*i*) by $x^2$, we have :

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

*i.e.* $1 + \left(\dfrac{y}{x}\right)^2 = \left(\dfrac{r}{x}\right)^2$

*i.e.* $\mathbf{1 + \tan^2\theta = \sec^2\theta}$

(*iii*) Dividing both sides of (*i*) by $y^2$, we get

$$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2}$$

*i.e.* $\left(\dfrac{x}{y}\right)^2 + 1 = \left(\dfrac{r}{y}\right)^2$

*i.e.* $\mathbf{\cot^2\theta + 1 = \mathrm{cosec}^2\theta}$

We have established the following fundamental identities :

(i) $\mathbf{\sin^2\theta + \cos^2\theta = 1}$

(ii) $\mathbf{\sec^2\theta = 1 + \tan^2\theta}$

(iii) $\mathbf{\mathrm{cosec}^2\theta = 1 + \cot^2\theta}$

These results can also be expressed as follows :

$$\sin^2\theta = 1 - \cos^2\theta$$
$$\cos^2\theta = 1 - \sin^2\theta$$

$$\tan^2\theta = \sec^2\theta - 1$$
$$\cot^2\theta = \operatorname{cosec}^2\theta - 1$$

Let us now solve some examples.

**Example 1 :** Given $\cos\theta = \frac{3}{5}$. Calculate $\sin\theta$, $\tan\theta$ and $\operatorname{cosec}\theta$.

**Solution :** We know that

$$\cos^2\theta + \sin^2\theta = 1$$

Substituting the value of $\cos\theta$, we get

$$\left(\frac{3}{5}\right)^2 + \sin^2\theta = 1$$

*i.e.* $$\sin^2\theta = 1 - \frac{9}{25} = \frac{16}{25} = \left(\frac{4}{5}\right)^2$$

$\therefore$ $$\sin\theta = \pm\frac{4}{5}$$

$\theta$ being acute and positive, minus sign is ignored.

$\therefore$ $$\sin\theta = \frac{4}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$$

**Example 2 :** Given $\cot\theta = \frac{20}{21}$, calculate the values of $\cos\theta$ and $\operatorname{cosec}\theta$.

**Solution :** Now $\cot\theta = \frac{20}{21}$

Here the ratio of adjacent side to opposite side is 20 : 21. So let $BC = 20a$ and $AB = 21a$, $a$ being the constant of proportionality (Fig. 13.6).

By Pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$
$$= (21a)^2 + (20a)^2 = 841a^2 = (29a)^2$$

$\therefore$ $AC = 29a$

$$\cos\theta = \frac{BC}{AC} = \frac{20a}{29a} = \frac{20}{29}$$

$$\operatorname{cosec}\theta = \frac{AC}{AB} = \frac{29a}{21a} = \frac{29}{21}$$

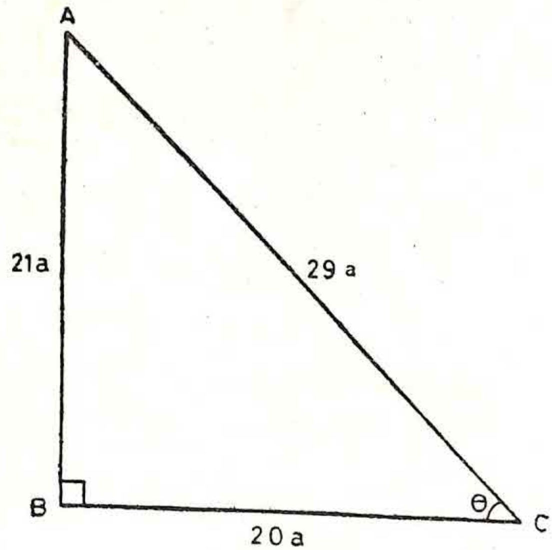



Fig. 13.6

**Example 3 :** Prove that

(*i*) $\sin^4 A - \cos^4 A = 1 - 2\cos^2 A$

(*ii*) $\cos^4 A + \sin^4 A + 2\sin^2 A\cos^2 A = 1$

**Solution :**

(*i*)
$$\begin{aligned}\sin^4 A - \cos^4 A &= (\sin^2 A)^2 - (\cos^2 A)^2\\ &= (\sin^2 A + \cos^2 A)(\sin^2 A - \cos^2 A)\\ &= (1)\,[(1 - \cos^2 A) - \cos^2 A]\\ &= 1 - 2\cos^2 A\end{aligned}$$

(*ii*)
$$\begin{aligned}\cos^4 A + \sin^4 A + 2\sin^2 A\cos^2 A &= (\cos^2 A)^2 + (\sin^2 A)^2 + 2\sin^2 A \times \cos^2 A\\ &= (\cos^2 A + \sin^2 A)^2\\ &= (1)^2\\ &= 1\end{aligned}$$

**Example 4 :** Prove that

(*i*) $$\sqrt{\frac{1+\sin A}{1-\sin A}} = \frac{1+\sin A}{\cos A} = \sec A + \tan A$$

(*ii*) If $\sin\theta = \frac{4}{5}$, then evaluate the following expression :

$$\frac{\cos\theta - \frac{1}{\cot\theta}}{2\operatorname{cosec}\theta}$$

**Solution :** (*i*)

$$\begin{aligned}\sqrt{\frac{1+\sin A}{1-\sin A}} &= \sqrt{\frac{1+\sin A}{1-\sin A}\times\frac{1+\sin A}{1+\sin A}}\\ &= \sqrt{\frac{(1+\sin A)^2}{1-\sin^2 A}}\\ &= \frac{1+\sin A}{\sqrt{\cos^2 A}}\\ &= \frac{1+\sin A}{\cos A}\\ &= \frac{1}{\cos A} + \frac{\sin A}{\cos A}\\ &= \sec A + \tan A\end{aligned}$$

(*ii*) Now, $\sin\theta = \frac{4}{5}$

$$\begin{aligned}\cos^2\theta &= 1 - \sin^2\theta\\ &= 1 - \frac{16}{25} = \frac{9}{25} = \left(\frac{3}{5}\right)^2\end{aligned}$$

$\therefore \quad \cos\theta = \frac{3}{5}$

$$\therefore \quad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

Now, $$\frac{\cos\theta - \frac{1}{\cot\theta}}{2\operatorname{cosec}\theta} = \frac{\cos\theta - \frac{1}{\cot\theta}}{\frac{2}{\sin\theta}}$$

$$= \frac{\frac{3}{5} - \frac{4}{3}}{\frac{5}{2}}$$

$$= \frac{-\frac{11}{15}}{\frac{5}{2}}$$

$$= -\frac{22}{75}$$

## Exercises 13.1

1. Given $\sin\theta = \frac{12}{13}$, find the values of $\tan\theta$ and $\cos\theta$.

2. Given $\tan\theta = \frac{20}{21}$, determine the values of $\cos\theta$ and $\sin\theta$.

3. Evaluate $\dfrac{\cos\theta - \frac{1}{\tan\theta}}{2\cot\theta}$ when $\sin\theta = \frac{3}{5}$

4. Given $\cos\theta = \frac{5}{13}$, determine the value of the expression

$$\frac{\frac{1}{\cos\theta}}{\sin\theta - \tan\theta}$$

5. Given $\cos\theta = \frac{p}{q}$, find the values of $\cot\theta$ and $\sin\theta$.

6. Prove that $\cos^4\theta + \sin^4\theta - 2\sin^2\theta\cos^2\theta = 1 - 4\cos^2\theta\sin^2\theta$.

Prove that :—

7. $\dfrac{\sin A}{1+\cos A}+\dfrac{\sin A}{1-\cos A}=\dfrac{2}{\sin A}$

8. $\sqrt{\dfrac{1+\cos A}{1-\cos A}}=\dfrac{1+\cos A}{\sin A}=\operatorname{cosec} A+\cot A$

9. $\sin^4 A-\cos^4 A=\sin^2 A-\cos^2 A=2\sin^2 A-1$

10. $\cos\theta\tan\theta+\cos\theta\sec\theta=1+\sin\theta$

11. $(\cos A-\sin A)^2+(\cos A+\sin A)^2=2$

12. $\sec A(1-\sin A)(\sec A+\tan A)=1$

13. $\dfrac{1+\cos A}{\sin A}+\dfrac{\sin A}{1+\cos A}=\dfrac{2}{\sin A}$

14. $\dfrac{\sec\theta+\tan\theta}{\sec\theta-\tan\theta}=(\sec\theta+\tan\theta)^2$

15. If $\tan\theta=\dfrac{a}{b}$, find the value of the expression $\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$.

(*Hint* : Divide the numerator and the denominator by $\cos\theta$.)

### 13.5 Behaviour of Trigonometric Ratios as $\theta$ Varies from 0° to 90°

Now we examine as to how the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ behave while $\theta$ varies from 0° to 90°.

Consider a point $P(x, y)$ on the arc of a unit circle (Fig. 13.7). We have taken the unit circle only for the sake of convenience. You can see by drawing the figure that the trigonometric ratio of the angle does not depend upon the radius of the circle on which $P$ is taken. Find the reason for this.

So the radius of circle $= OP = 1$

From $P$, draw $PM$ perpendicular on $x$-axis and denote $\angle MOP$ by $\theta$.

Let $P$ be a moving point taking different positions on the arc of the circle starting from the point $R$ on the $x$-axis.

In its initial position,

$\angle MOP = 0°$.

In this case, the abscissa of the point $P$ is the radius of the circle *i.e.* 1 and the ordinate is zero.

Similarly in another position, let the point $P$ be on the $y$-axis coinciding with $S$, resulting in $\angle MOP = 90°$. In this case, abscissa of the point $P$ is zero and the ordinate is 1.

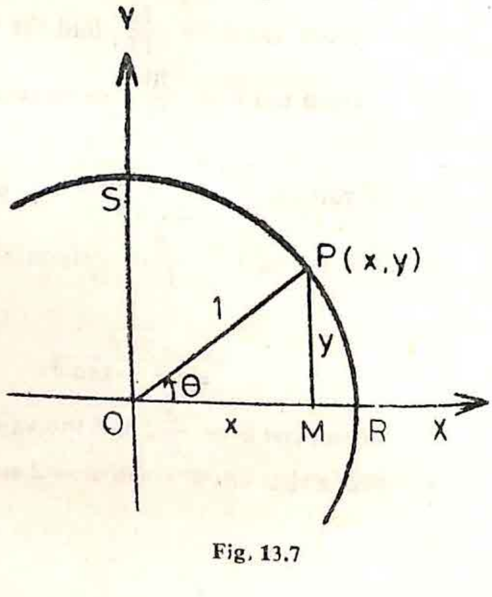


Fig. 13.7

Now we observe the following:

When $\theta = 0°$, $y = 0$ and $x = 1$

$$\therefore \quad \sin 0° = \frac{y}{r} = \frac{y}{1} = 0$$

$$\cos 0° = \frac{x}{r} = \frac{x}{1} = 1$$

$$\tan 0° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0$$

Again when $\theta = 90°$, $y = 1$ and $x = 0$

$$\therefore \quad \sin 90° = \frac{y}{r} = \frac{y}{1} = \frac{1}{1} = 1$$

$$\cos 90° = \frac{x}{r} = \frac{x}{1} = 0$$

$$\tan 90° = \frac{\sin 90°}{\cos 90°}$$

Since cos 90° is zero and division by zero is not defined, **tan 90° is not defined.**

In Fig. 13.8, let $\angle POM = \theta$ and as $P$ takes different positions, $\theta$ varies from 0° to 90°.

You see in Fig. 13.8 that the perpendicular $P_1M_1$ of $\triangle P_1OM_1$ is greater than the perpendicular $PM$ of $\triangle POM$.

So with the increase in the angle, the perpendicular *i.e.* opposite side goes on increasing while the adjacent side of the triangle goes on decreasing.

Therefore, when angle $\theta$ increases from 0° to 90°, the sine of the angle *i.e.* $\sin \theta$ also increases. At $\theta = 90°$ it is equal to 1. So $\sin \theta$ increases from 0 to 1 as $\theta$ increases from 0° to 90°.

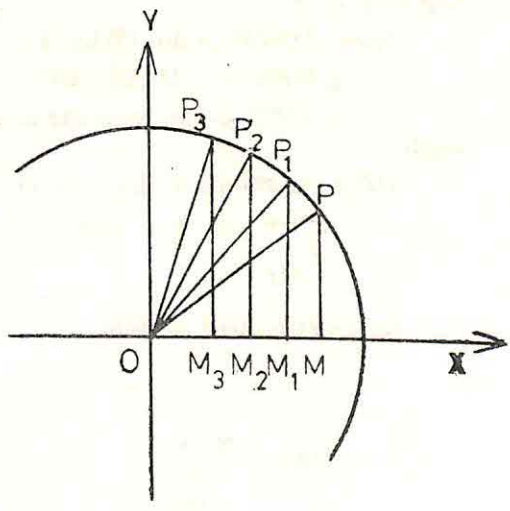


Fig. 13.8

Similarly, the cosine of the angle decreases from 1 to 0 as $\theta$ increases from 0° to 90° because the adjacent side goes on decreasing.

Now we can observe the behaviour of sin $\theta$, cos $\theta$ and tan $\theta$ from the following **tabular form :**

| *At $\theta = 0°$* | *As $\theta$ varies from 0° to 90°* | *At $\theta = 90°$* |
|---|---|---|
| $y = 0$ | $y$ increases from 0 to 1 | $y = 1$ |
| $x = 1$ | $x$ decreases from 1 to 0 | $x = 0$ |
| $\sin\theta = 0$ | $\sin\theta$ increases from 0 to 1 | $\sin\theta = 1$ |
| $\cos\theta = 1$ | $\cos\theta$ decreases from 1 to 0 | $\cos\theta = 0$ |
| $\tan\theta = 0$ | $\tan\theta$ increases | $\tan\theta$ is not defined |

## 13.6 Trigonometrical Ratios for Some Angles

### *A. Trigonometrical Ratios for an Angle of 30°*

**Consider** the position of the point $P$ on **the unit circle** such that $\angle ROP = 30°$.

From $P$, draw $PM \perp x$-axis and **produce** it to cut the circle at $Q$. Join $OQ$ (Fig. 13.9) .

Now, $\angle QOM = 30°$ (Why ?)

$\therefore \quad \angle MPO = \angle MQO = 60°$

$\therefore \quad \triangle\ OPQ$ is an equilateral triangle.

$OP$ is the radius of the unit circle.

$\therefore \quad PQ = OP = 1$ and

$$MP = \frac{1}{2}$$

So $OM^2 = OP^2 - MP^2$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

$$\therefore \quad OM = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \sin 30° = \frac{MP}{1}$$

$$= \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

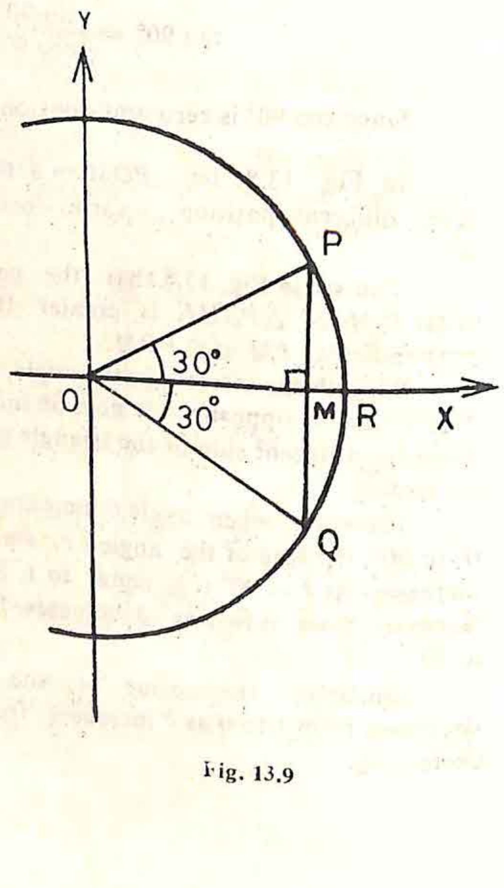



Fig. 13.9

$$\cos 30^\circ = \frac{OM}{1} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{MP}{OM} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{OM}{MP} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$\sec 30^\circ = \frac{1}{OM} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

$$\text{cosec } 30^\circ = \frac{1}{MP} = \frac{1}{\frac{1}{2}} = 2$$

*B. Trigonometrical Ratios for an Angle of 60°*

Consider the position of the point $P$ on the unit circle such that

$\angle QOP = 60^\circ$ (Fig. 13.10).

Draw $PM \perp OQ$. Join $PQ$.

Now $OP = OQ = 1$.

$MP$ is the right bisector of $OQ$. (Why ?)

Therefore, $OM = \frac{1}{2}$

Now, $MP^2 = OP^2 - OM^2$

$= 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore \quad MP = \frac{\sqrt{3}}{2}$

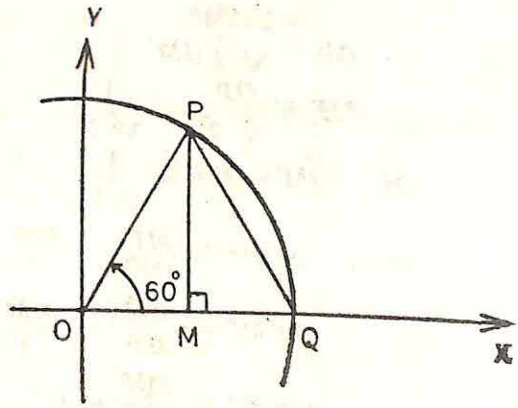



Fig. 13.10

Now, $\sin 60^\circ = \frac{MP}{OP} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$

$$\cos 60^\circ = \frac{OM}{OP} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{MP}{OM} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$\cot 60^\circ = \frac{OM}{MP} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$$\sec 60^\circ = \frac{OP}{OM} = \frac{1}{\frac{1}{2}} = 2$$

$$\operatorname{cosec} 60^\circ = \frac{OP}{MP} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

*C. Trigonometrical Ratios for an Angle of 45°*

Consider the position of the point $P$ on the unit circle such that $\angle QOP = 45°$. From $P$, draw $PM \perp OX$ (Fig. 13.11). Here we observe that $MOP$ is an isosceles triangle and

$$OM = MP$$

$$\therefore \quad OP^2 = OM^2 + MP^2$$
$$= OM^2 + OM^2$$
$$= 2\,OM^2$$

$$\therefore \quad OP = \sqrt{2}\,OM$$

$$\therefore \quad OM = \frac{OP}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

So, $\quad OM = MP = \dfrac{1}{\sqrt{2}}$

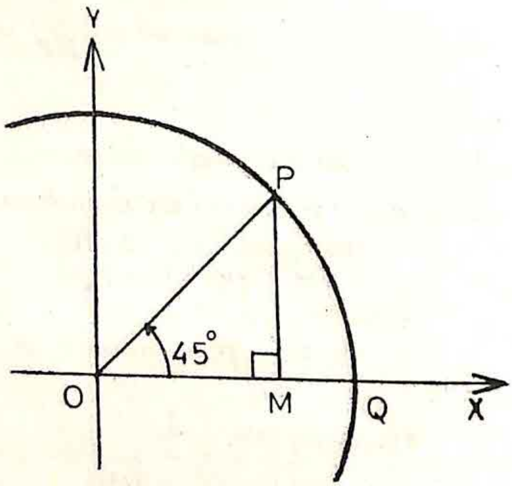


Fig. 13.11

Now, $\quad \sin 45^\circ = \dfrac{MP}{OP} = \dfrac{MP}{1} = \dfrac{1}{\sqrt{2}}$

$$\cos 45^\circ = \frac{OM}{OP} = \frac{OM}{1} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{MP}{OM} = 1$$

$$\cot 45^\circ = \frac{OM}{MP} = 1$$

$$\sec 45^\circ = \frac{OP}{OM} = \frac{1}{OM} = \sqrt{2}$$

$$\operatorname{cosec} 45^\circ = \frac{OP}{MP} = \frac{1}{MP} = \sqrt{2}$$

*D. Trigonometrical Ratios for an Angle of 0°*

We have already seen in section 13.5 that $\sin 0° = 0$, $\cos 0° = 1$ and $\tan 0° = 0$.

As for other trigonometrical ratios are concerned, we have

$$\sec 0° = \frac{1}{\cos 0°} = \frac{1}{1} = 1$$

Now $\cot 0° = \frac{1}{\tan 0°}$. We know that $\tan 0° = 0$. Since division by zero is not defined, $\cot 0°$ is not defined. Similarly $\text{cosec } 0° = \frac{1}{\sin 0°}$, $\sin 0° = 0$ and hence $\text{cosec } 0°$ is also not defined.

*E. Trigonometrical Ratios for an Angle of 90°*

In section 13.5 we established that the values of $\sin 90° = 1$ and $\cos 90° = 0$.

As discussed earlier, $\tan 90°$ is not defined.

Similarly, $\sec 90°$ is not defined,

while $\cot 90° = \frac{\cos 90°}{\sin 90°} = \frac{0}{1} = 0$

$$\text{cosec } 90° = \frac{1}{\sin 90°} = \frac{1}{1} = 1$$

Now we summarize the values of the trigonometrical ratios in the following tabular form:

| T-ratio \ Angle $\theta$ | 0° | 30° | 45° | 60° | 90° |
|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| $\tan\theta$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | Not defined |
| $\cot\theta$ | Not defined | $\sqrt{3}$ | 1 | $\frac{1}{\sqrt{3}}$ | 0 |
| $\sec\theta$ | 1 | $\frac{2}{\sqrt{3}}$ | $\sqrt{2}$ | 2 | Not defined |
| $\text{cosec }\theta$ | Not defined | 2 | $\sqrt{2}$ | $\frac{2}{\sqrt{3}}$ | 1 |

An aid to memory for writing the values of sin $\theta$ and cos $\theta$ for the above angles is as under :

| Angle $\theta$ / T-ratio | 0° | 30° | 45° | 60° | 90° |
|---|---|---|---|---|---|
| sin $\theta$ | $\sqrt{\frac{0}{4}}$ | $\sqrt{\frac{1}{4}}$ | $\sqrt{\frac{2}{4}}$ | $\sqrt{\frac{3}{4}}$ | $\sqrt{\frac{4}{4}}$ |
| cos $\theta$ | $\sqrt{\frac{4}{4}}$ | $\sqrt{\frac{3}{4}}$ | $\sqrt{\frac{2}{4}}$ | $\sqrt{\frac{1}{4}}$ | $\sqrt{\frac{0}{4}}$ |

**13.7 Trigonometrical Ratios of Complementary Angles**

Let us consider the right triangle $QOP$ (Fig. 13.12).

Let $\angle QOP = \theta$.

In $\triangle QOP$, $OQ$ is the adjacent side and $QP$ is the opposite side with respect to $\angle QOP$.

So,

$$\sin\theta = \frac{PQ}{OP}, \qquad \operatorname{cosec}\theta = \frac{OP}{PQ}$$

$$\cos\theta = \frac{OQ}{OP}, \qquad \sec\theta = \frac{OP}{OQ}$$

$$\tan\theta = \frac{PQ}{OQ}, \qquad \cot\theta = \frac{OQ}{PQ}$$

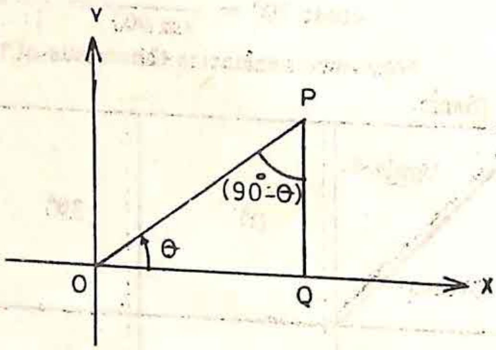



Fig. 13.12

Considering the angle $QPO$ which is $90° - \theta$, $QP$ is the adjacent side and $OQ$ is the opposite side.

So we have :

$$\sin(90° - \theta) = \frac{OQ}{OP} = \cos\theta$$

$$\cos(90° - \theta) = \frac{QP}{OP} = \sin\theta$$

$$\tan(90° - \theta) = \frac{OQ}{QP} = \cot\theta$$

$$\cot(90° - \theta) = \frac{QP}{OQ} = \tan\theta$$

$$\sec(90° - \theta) = \frac{OP}{QP} = \operatorname{cosec}\theta.$$

$$\operatorname{cosec}(90° - \theta) = \frac{OP}{OQ} = \sec\theta$$

**Example 1 :** Evaluate $\dfrac{\cos 70^\circ}{\sin 20^\circ}$.

**Solution :** $\cos 70^\circ = \cos(90^\circ - 20^\circ) = \sin 20^\circ$

$$\therefore \quad \frac{\cos 70^\circ}{\sin 20^\circ} = \frac{\sin 20^\circ}{\sin 20^\circ} = 1$$

**Example 2 :** Find the value of

(*i*) $\dfrac{\sin 30^\circ}{\cos 45^\circ} + \dfrac{\tan 45^\circ}{\sec 60^\circ} - \dfrac{\sin 60^\circ}{\cot 45^\circ} - \dfrac{\cos 30^\circ}{\sin 90^\circ}$

(*ii*) $\sin^2 60^\circ + \cos^2 30^\circ + \tan^2 45^\circ + \sec^2 60^\circ - \operatorname{cosec}^2 30^\circ$

**Solution :** (*i*) $\dfrac{\sin 30^\circ}{\cos 45^\circ} + \dfrac{\tan 45^\circ}{\sec 60^\circ} - \dfrac{\sin 60^\circ}{\cot 45^\circ} - \dfrac{\cos 30^\circ}{\sin 90^\circ}$

$$= \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} + \frac{1}{2} - \frac{\frac{\sqrt{3}}{2}}{1} - \frac{\frac{\sqrt{3}}{2}}{1}$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = \frac{\sqrt{2} + 1 - \sqrt{3} - \sqrt{3}}{2}$$

$$= \frac{\sqrt{2} + 1 - 2\sqrt{3}}{2}$$

(*ii*) $\sin^2 60^\circ + \cos^2 30^\circ + \tan^2 45^\circ + \sec^2 60^\circ - \operatorname{cosec}^2 30^\circ$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + (1)^2 + (2)^2 - (2)^2$$

$$= \frac{3}{4} + \frac{3}{4} + 1 + 4 - 4$$

$$= \frac{3}{2} + 1 = \frac{5}{2}$$

## Exercises 13.2

1. Calculate the value of each of the following :

(*i*) $1 + \tan^2 60^\circ$

(*ii*) $\cot^2 45^\circ + \cos^2 30^\circ$

(*iii*) $1 - \sec^2 30^\circ$

(*iv*) $1 + \operatorname{cosec}^2 30^\circ$

(*v*) $\cos^2 0^\circ - \tan^2 45^\circ$

(*vi*) $\dfrac{\sin 49^\circ}{\cos 41^\circ}$

(*vii*) $\dfrac{\tan 59^\circ}{\cot 31^\circ}$

2. Find the value of

(i) $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$

(ii) $\dfrac{\tan 45^\circ}{\sin 30^\circ + \cos 30^\circ}$

(iii) $\dfrac{\tan 60^\circ}{\sec 60^\circ + \operatorname{cosec} 60^\circ}$

3. Find the value of

$\tan^2 60^\circ + 4\cos^2 45^\circ + 3\sec^2 30^\circ + 5\cos^2 90^\circ$

4. Prove that

(i) $2\sin^2 60^\circ \cos 60^\circ = \dfrac{3}{4}$ (ii) $\dfrac{\operatorname{cosec} 39^\circ}{\sec 51^\circ} = 1$

**13.8 Miscellaneous Examples Involving Trigonometrical Ratios**

A list of formulae and identities is given below for ready reference:

(a) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$; $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(b) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$; $\sec\theta = \dfrac{1}{\cos\theta}$; $\cot\theta = \dfrac{1}{\tan\theta}$

(c) $\sin^2\theta + \cos^2\theta = 1$; $\sec^2\theta = 1 + \tan^2\theta$; $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

(d) $\sin(90^\circ - \theta) = \cos\theta$; $\cos(90^\circ - \theta) = \sin\theta$; $\tan(90^\circ - \theta) = \cot\theta$;
$\operatorname{cosec}(90^\circ - \theta) = \sec\theta$; $\sec(90^\circ - \theta) = \operatorname{cosec}\theta$; $\cot(90^\circ - \theta) = \tan\theta$

**Example 1 :** Prove that $\left(\dfrac{\sec\theta + 1}{\tan\theta}\right)^2 = \dfrac{\sec\theta + 1}{\sec\theta - 1}$

**Solution :**

$$\left(\frac{\sec\theta + 1}{\tan\theta}\right)^2 = \frac{(\sec\theta + 1)^2}{\tan^2\theta}$$
$$= \frac{(\sec\theta + 1)^2}{\sec^2\theta - 1}$$
$$= \frac{(\sec\theta + 1)^2}{(\sec\theta + 1)(\sec\theta - 1)}$$
$$= \frac{\sec\theta + 1}{\sec\theta - 1}.$$

**Example 2 :** Prove that $\cos^2\theta(1 + \tan^2\theta) + \sin^2\theta(1 + \cot^2\theta) = 2$

**Solution** · $\cos^2\theta(1 + \tan^2\theta) + \sin^2\theta(1 + \cot^2\theta)$

$$= \cos^2\theta\sec^2\theta + \sin^2\theta\operatorname{cosec}^2\theta$$
$$= \cos^2\theta . \frac{1}{\cos^2\theta} + \sin^2\theta \cdot \frac{1}{\sin^2\theta}$$
$$= \quad 1 \quad + \quad 1$$
$$= 2$$

**Example 3 :** Prove that $\dfrac{1}{1 - \sin\theta} + \dfrac{1}{1 + \sin\theta} = 2\sec^2\theta$

**Solution :** $\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}=\frac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)}$

$$=\frac{2}{1-\sin^2\theta}$$

$$=\frac{2}{\cos^2\theta}$$

$$=2\sec^2\theta$$

**Example 4 :** Verify that

$$4(\sin^4 60^\circ+\cos^4 30^\circ)-3(\tan^2 60^\circ-\tan^2 45^\circ)+5\cos^2 45^\circ=1$$

**Solution :** We know that

$\sin 60^\circ=\frac{\sqrt{3}}{2}$, $\cos 30^\circ=\frac{\sqrt{3}}{2}$, $\tan 60^\circ=\sqrt{3}$,

$\tan 45^\circ=1$ and $\cos 45^\circ=\frac{1}{\sqrt{2}}$. Substituting these values, we get

$$4(\sin^4 60^\circ+\cos^4 30^\circ)-3(\tan^2 60^\circ-\tan^2 45^\circ)+5\cos^2 45^\circ$$

$$=4\left[\left(\frac{\sqrt{3}}{2}\right)^4+\left(\frac{\sqrt{3}}{2}\right)^4\right]-3(\sqrt{3})^2+3(1)^2+5\left(\frac{1}{\sqrt{2}}\right)^2$$

$$=4\left(\frac{9}{16}+\frac{9}{16}\right)-9+3+\frac{5}{2}$$

$$=\frac{9}{2}-9+3+\frac{5}{2}$$

$$=1$$

## Exercises 13.3

Prove the following identities :

1. (*i*) $\frac{\cos\theta\,.\,\cot\theta}{1-\sin\theta}=1+\operatorname{cosec}\theta$

(*ii*) $\frac{1+\cos\theta}{1-\cos\theta}=\frac{\tan^2\theta}{(\sec\theta-1)^2}$

(*iii*) $\frac{\sec\theta+\tan\theta}{\sec\theta-\tan\theta}=(\sec\theta+\tan\theta)^2$

(*iv*) $\frac{\cos^2\theta}{\sin^2\theta}+1=\frac{\cot^2\theta}{\cos^2\theta}$

2. (*i*) $\frac{\cot\theta}{\frac{\cos^3\theta}{\sin\theta}+\sin\theta\cos\theta}=1$

(*ii*) $\dfrac{\sin^2 A}{\cos^2 A} + \dfrac{\cos^2 A}{\sin^2 A} = \dfrac{1}{\sin^2 A \cos^2 A} - 2$

3. Find the values of the expressions :
   (*i*) $\cos 60° \cos 45° - \sin 60° \sin 45°$
   (*ii*) $\sin 45° \cos 30° - \cos 45° \sin 30°$

4. By taking $A = 30°$, verify that
   (*i*) $\cos 2A = 2 \cos^2 A - 1$
   (*ii*) $\sin 2A = 2 \sin A \cos A$
   (*iii*) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

5. Find the value of
   (*i*) $\dfrac{4}{\cot^2 30°} + \dfrac{1}{\sin^2 60°} - \cos^2 45°$
   (*ii*) $3 \cos^2 60° + 2 \cot^2 30° - 5 \sin^2 45°$

6. If $\tan \theta = \dfrac{12}{13}$, find the value of
   (*i*) $\dfrac{\sin \theta + \cos \theta}{\cos \theta - \sin \theta}$
   (*ii*) $\dfrac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}$

CHAPTER 14

# Heights and Distances

## 14.1 Trigonometrical Table

In the previous chapter, we found the values of the trigonometrical ratios of certain angles such as 0°, 30°, 45°, 60° and 90°, using certain geometrical constructions. For finding the values of trigonometrical ratios of any angle between 0° and 90° table of trigonometrical ratios is available. (The table is printed as appendix in the end.) In the table the angles are listed at intervals of 10′. The values of the trigonometrical ratios given in the table are approximate, but they are adequate for all practical purposes. We have already seen that there are certain trigonometrical ratios which are not defined for certain angles such as tan 90°, sec 90°, cot 0° and cosec 0°. So the corresponding entries are left blank in the table. You will also notice that the table contains trigonometrical ratios of only sine(sin), cosine(cos) and tangent(tan) of angles. But you know that the other three trigonometrical ratios namely cosecant, secant and cotangent of angles are the reciprocals of the three given in the table and hence their values can be easily computed. Now let us understand how to refer to the table for finding the values of trigonometrical ratios of different angles.

A part of the table is reproduced below:

| *Degrees* | *sin* | *cos* | *tan* |
|---|---|---|---|
| 38°00′ | .6157 | .7880 | .7812 |
| 10 | .6180 | .7862 | .7860 |
| 20 | .6202 | .7844 | .7907 |
| 30 | .6225 | .7826 | .7954 |
| 40 | .6248 | .7808 | .8002 |
| 50 | .6271 | .7790 | .8050 |

Let us find, say, sin 38° 30′ and cos 38° 50′ from the table. The first column of the table gives the angles. We are to find sin 38° 30′. So, find where 38° 30′ is in the first column. It is in the *fourth row*. 'sin' is in the *second column*. The entry in the fourth row and second column, therefore, gives the value of sin 38° 30′. You find that sin 38° 30′ = .6225. In the same way, you can find cos 38° 50′ by looking into the entry in the sixth row (corresponding to the angle 38° 50′) and the third column (corresponding to the trigonometric ratio 'cos') and get cos 38° 50′ = .7790.

We can use the table not only for finding the values of the trigonometrical ratios of given angles, but, given the value of a trigonometrical ratio of an angle we can find the angle. For example, it is given that $\tan\theta = .7954$ and we are to find the value of $\theta$ (which is between 0° and 90°). We select the 'tan' column first and find where the given value of $\tan\theta$ is located. We find that it is in the fourth row. Now we will find the angle corresponding to this fourth row. The angle is 38° 30′. So the required angle is 38° 30′.

Sometimes we require the values of trigonometrical ratios of angles that are not multiples of 10′ and therefore they are not available in the table. In such cases we may *proceed* as follows. We may assume that for small variations in $\theta$, the variation in the trigonometrical ratio is proportional to the variation in $\theta$. For example, let us see how to find the value of sin 38° 38′ and cos 38° 38′.

From the table, we have :

$$\sin 38^\circ 30' = .6225$$
$$\sin 38^\circ 40' = .6248$$

$\therefore$ Difference for 10′ variation $= .6248 - .6225 = .0023$

$\therefore$ Difference for 8′ variation $= \dfrac{.0023}{10} \times 8 = .0018$

$$\therefore \quad \sin 38^\circ 38' = .6225 + .0018 = .6243$$

(We are adding .0018 to the value of sin 38° 30′, since the value of $\sin\theta$ increases as $\theta$ increases.)

Again from the table :

$$\cos 38^\circ 30' = .7826$$
$$\cos 38^\circ 40' = .7808$$

$\therefore$ Difference for 10′ variation $= .7826 - .7808 = .0018$

$\therefore$ Difference for 8′ variation $= \dfrac{.0018}{10} \times 8 = .0014$

$$\therefore \quad \cos 38^\circ 38' = .7826 - .0014 = .7812$$

(We are subtracting .0014 from the value of cos 38° 30′ since the value of $\cos\theta$ decreases as $\theta$ increases.)

**Example 1 :** Find the values of sin 47° 20′, cos 47° 10′ and tan 47° 50′ from the table.

**Solution :** We look into the table given in the appendix.

We find that:

$$\sin 47^\circ 20' \doteq .7353$$
$$\cos 47^\circ 10' = .6799$$
$$\tan 47^\circ 50' = 1.104$$

Example 2 : Find the values of $x, y, z$ (between $0^\circ$ and $90^\circ$) given that :

$$\tan x = 1.643$$
$$\cos y = .5250$$
$$\sin z = .7528$$

**Solution :** We look into the table given in the appendix.

We look for the entry 1.643 in the 'tan' column and look for the angle in the corresponding row. We find that $\tan 58^\circ 40' = 1.643$

$\therefore \quad x = 58^\circ 40'$

In the same way, we find that :

$$y = 58^\circ 20'$$
$$z = 48^\circ 50'$$

## Exercises 14.1

1. Find the value of each of the following trigonometrical ratios :

   (*i*) $\sin 61^\circ$ (*ii*) $\cos 53^\circ$ (*iii*) $\tan 53^\circ 10'$
   (*iv*) $\cos 39^\circ 50'$ (*v*) $\sin 7^\circ 40'$ (*vi*) $\cos 1^\circ 30'$
   (*vii*) $\cos 44^\circ 38'$ (*viii*) $\tan 67^\circ 35'$

2. For each of the following values, find the corresponding value of $\theta$ :

   (*i*) $\sin \theta = .2504$ (*ii*) $\cos \theta = .9159$ (*iii*) $\tan \theta = 1.079$
   (*iv*) $\sin \theta = .8465$ (*v*) $\cos \theta = .4462$ (*vi*) $\tan \theta = 2.651$
   (*vii*) $\sin \theta = .7001$ (*viii*) $\cos \theta = .3781$

### 14.2 Heights and Distances

Fig. 14.1 shows a flag fluttering on the top of a pole $AC$. We are interested to find the height of this pole above the ground. How can we find the height ?

Let us take a point $B$ on the ground. We can easily measure $BC$. Now $AC$ subtends $\angle ABC$ at $B$. If we can measure this angle $ABC$, then we can find $AC$ with the help of our knowledge of trigonometry.

Imagine you are at $B$ (as a point) and looking up at $A$. Your line of sight is along $BA$ and it is making $\angle ABC$ with the horizontal line $BC$. This angle $ABC$ is called the *angle of elevation* of $A$ from $B$.

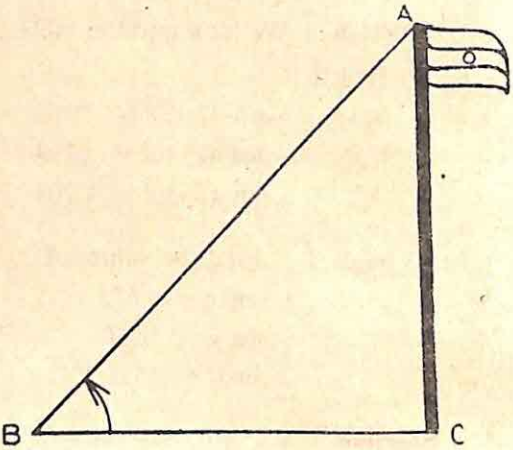



Fig. 14.1

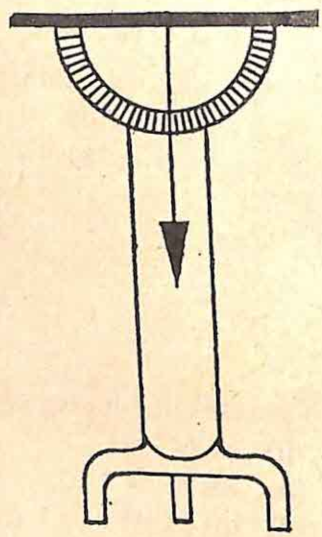
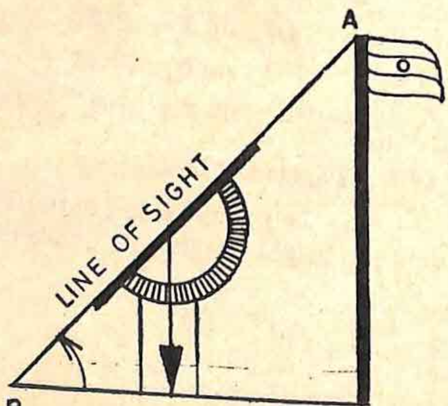


Fig. 14.2

Now how to measure the angle of elevation i.e. $\angle ABC$ ? A simple instrument known as CLINOMETER is used for this purpose (Fig. 14.2). It consists of a vertical stand on which a protractor is hinged. A plumb line is freely suspended from the centre. The protractor can be rotated about the hinge.

To find the angle of elevation $ABC$, the clinometer is placed at a convenient point (Fig. 14.3). Keeping the eye at $B$, the protractor is rotated such that the line of sight $BA$ coincides with straight edge of the protractor. The reading on the protractor as indicated by the plumb line (which always remains vertical) gives the angle of elevation.



Suppose you are on the top of a tower $AC$ and looking down at an object $B$ on the ground (Fig. 14.4). Your line of sight will be along $AB$. This line of sight $AB$ makes angle $DAB$ with the horizontal through $A$. This angle $DAB$ is called the *angle of depression* of $B$ from $A$. You can easily see that the angle of elevation of $A$ from $B$ is equal to the angle of depression of $B$ from $A$. Explain 'why'.

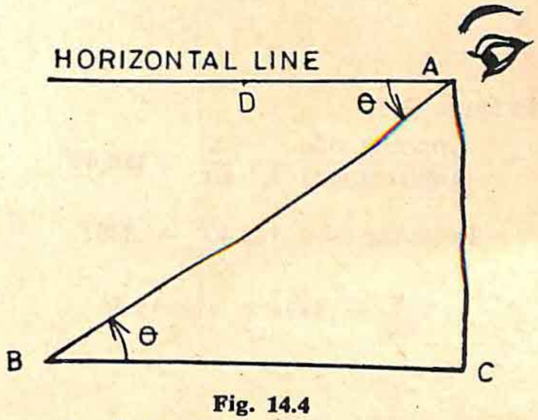



**Fig. 14.4**

It is stated above that once we know the angle of elevation and the distance of the point of observation from the foot of the tower, we can find the height of the tower using the knowledge of trigonometry. The following examples illustrate the method.

**Example 1 :** In the right triangles of figures 14.5 to 14.7, $x$ denotes the length of the side. Find the value of $x$.

**Solution :**

In figure 14.5,

$$\frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{x}{25} = \sin 30°$$

We know that $\sin 30° = \frac{1}{2}$

$$\therefore \frac{x}{25} = \frac{1}{2} \Rightarrow x = 12.5$$

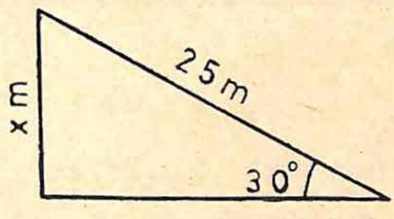



**Fig. 14.5**

In figure 14.6,

$$\frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{x}{35} = \cos 60°$$

We know that $\cos 60° = \frac{1}{2}$

$$\therefore \frac{x}{35} = \frac{1}{2} \Rightarrow x = 17.5$$

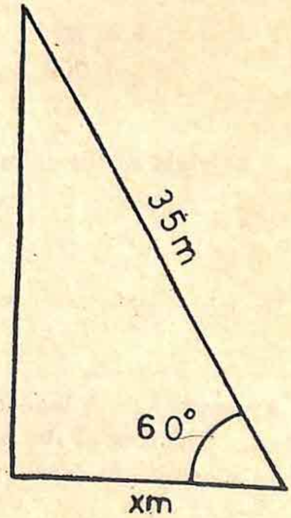



**Fig. 14.6**

In figure 14.7,

$$\frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{x}{40} = \tan 40^\circ$$

From the table, $\tan 40^\circ = .8391$

$$\therefore \frac{x}{40} = .8391 \Rightarrow x = 33.56$$

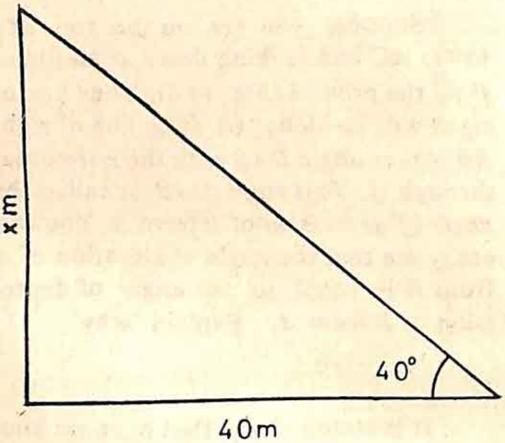



Fig. 14.7

**Example 2 :** The angle of elevation of the top of a chimney $AB$ is $58^\circ$ from a point $C$ on the ground at a distance of 200 metres from the base of the chimney. Calculate the height of the chimney.

**Solution :** Let the height of the chimney be $x$ metres (Fig. 14.8).

In $\triangle ABC$,

$$\frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{x}{200} = \tan 58^\circ$$

$$\begin{aligned} \therefore \quad x &= 200 \times \tan 58^\circ \\ &= 200 \times 1.60 \\ &= 320 \end{aligned}$$

$\therefore$ Height of the chimney is 320 m.

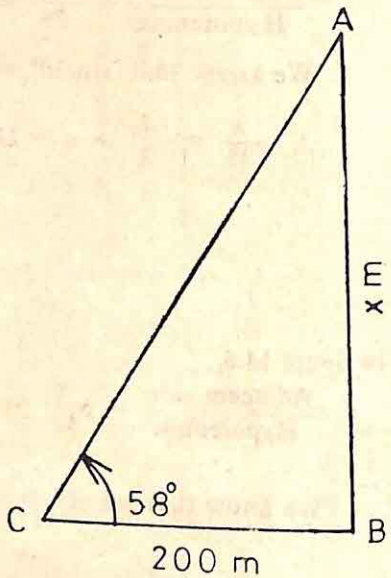



Fig. 14.8

**Example 3 :** A ladder is placed against a wall such that it just reaches the top of the wall. The foot of the ladder is 1.5 metres away from the wall and the ladder is inclined at an angle of $68^\circ$ with the ground. Find the height of the wall.

**Solution :** Let the height of wall be $h$ m (Fig. 14.9).

In $\triangle$ $ACB$,

$$\frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{h}{1.5} = \tan 68^\circ$$

$$\therefore \quad h = 1.5 \times \tan 68^\circ$$
$$= 1.5 \times 2.475$$
$$= 3.71 \text{ (ayprox.)}$$

$\therefore$ Height of the wall = 3.71 m

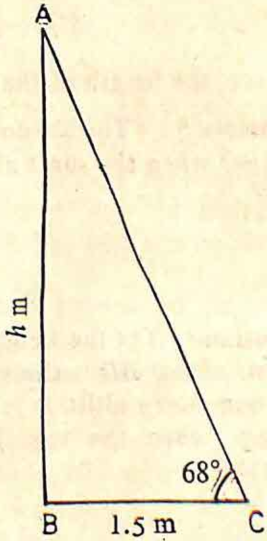



Fig. 14.9

**Example 4 :** An electric pole is 10 m high. A steel wire tied to the top of the pole and affixed at a point on the ground to keep the pole vertical, makes an angle of 49° with the horizontal line through the foot of the pole. Find the length of the steel wire.

**Solution :** Let $AB$ be the pole and let the length of the steel wire $AC$ be $x$ m (Fig. 14.10).

In $\triangle$ $ACB$,

$$\frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{10}{x} = \sin 49^\circ$$

$$\therefore \quad \frac{10}{x} = .7547$$

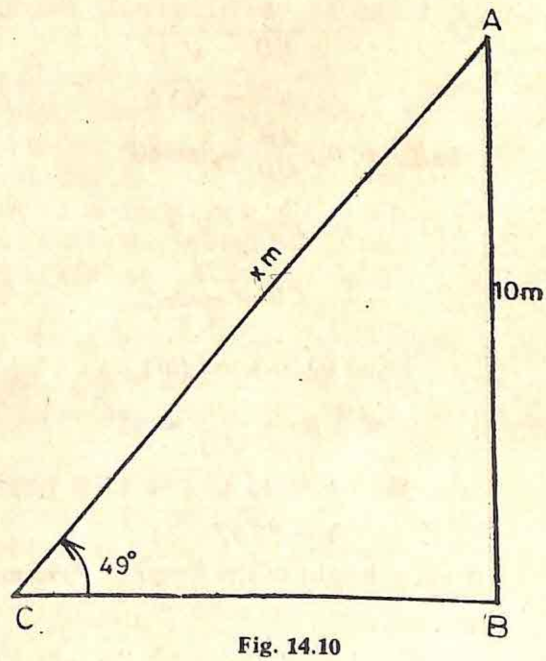



Fig. 14.10

$$\therefore \qquad x = \frac{10}{.7547} = 13.25$$

Hence, the length of the wire = 13.25 m

**Example 5 :** The shadow of a tower standing on a level ground is found to be 45 m longer when the sun's altitude is 30° than when it was 60°. Find the height of the tower.

**Solution :** Let the height of the tower $AB$ be $h$ m. Now $BD$ is the shadow of the tower when sun's altitude is 30° and $BC$ is the shadow when the sun's altitude is 60° (Fig. 14.11).

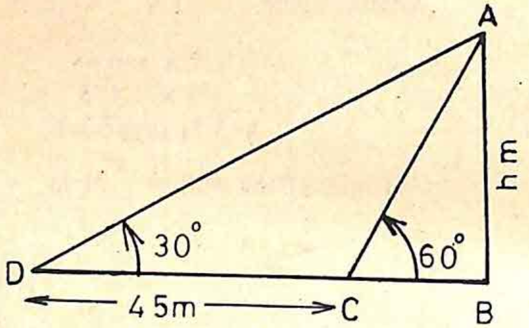



**Fig. 14.11**

Now $DC = 45$ m, i.e. $BD - CB = 45$ m ...(*i*)

In $\triangle ABD$, $\dfrac{AB}{BD} = \tan 30°$

*i.e.* $\dfrac{h}{BD} = \dfrac{1}{\sqrt{3}}$

$\therefore \quad BD = \sqrt{3}\,h$ ...(*ii*)

In $\triangle ACB$, $\dfrac{AB}{CB} = \tan 60°$

*i.e.* $\dfrac{h}{CB} = \sqrt{3}$

$\therefore \quad CB = \dfrac{h}{\sqrt{3}}$ ...(*iii*)

$\therefore$ From (*i*), (*ii*) and (*iii*),

$$\sqrt{3}\,h - \frac{h}{\sqrt{3}} = 45$$

*i.e.* $3h - h = 45\sqrt{3} = 45 \times 1.732$

$\therefore \quad h = 38.97$

Hence, height of the tower is 38.97 m.

## Exercises 14.2

1. A tower subtends an angle of 30° at a point on the ground whose distance is 150 metres from its base. Find the height of the tower.
2. The angles of depression of two ships on either side of a light-house as observed from the top of the light-house are 52° and 41°40′ respectively. If the height of the light-house is 150 metres, find the distance between the two ships.
3. A vertical pole is 300 metres high. Find the angle subtended by the pole at a point $300\sqrt{3}$ metres from its base.
4. The shadow of a minar is 80 metres long when the altitude of the sun is 41°50′. Find the height of the minar.
5. A ladder resting against a vertical wall, is inclined at an angle of 29° to the ground. The foot of the ladder is 2.5 metres from the wall. Find the length of the ladder.
6. Two ships are sailing in the sea on either side of a light-house. The angles of depression of two ships as observed from the top of the light-house are 50° and 40°30′ respectively. If the light-house is 200 metres high, find the distance between the ships.
7. A helicopter, at an altitude of 1500 metres, finds that two ships are sailing towards it, in the same direction. The angles of depression of the ships as observed from the helicopter are 60° and 30° respectively. Find the distance between the two ships.
8. Two poles of equal heights are standing opposite to each other on either side of a road, which is 100 metres wide. From a point between them on the road, the angles of elevation of the tops are 30° and 60°. Find the position of the point and also the heights of the poles.
9. A flag-staff stands on the top of a 5 m high tower. From a point on the ground the angle of elevation of the top of the flag-staff is 60° and from the same point the angle of elevation of the top of the tower is 45°. Find the height of the flag-staff.
10. A pole 5 m high is fixed on the top of a tower. The angle of elevation of the top of the pole observed from a point $A$ on the ground is 57°. The angle of depression of the point $A$ from the top of the tower is 42°. Find the height of the tower.
11. On the same side of a tower two objects are located. Observed from the top of the tower, their angles of depression are 45° and 60°. Find the distance between the two objects, if the height of the tower is 300 metres.
12. A kite is flown with a thread 250 metres long. If the thread is assumed stretched straight and makes an angle of 55° with the horizontal, find the height of the kite above the ground.

CHAPTER 15

# Surface Area and Volume

## 15.1 Solid Figures

In the earlier classes, we have studied about area of plane figures. Let us recall that area is the measure of the region enclosed by a plane figure. We will now consider some figures other than plane figures which occupy space and have more than two dimensions. Figures having three dimensions are called solids. A box, a jar, a ball, an inkpot, a brick are some examples of solids.

## 15.2 Cuboid and Cube

A solid figure bounded by six rectangular plane surfaces (faces) is called a cuboid. A chalk box, a brick or a matchbox, are examples of a cuboid. All the three dimensions of a cuboid need not be the same. Fig. 15.1 is the diagram of a cuboid. A cuboid has three pairs of parallel and congruent faces. Any two adjacent faces meet along a line segment called *edge* of the cuboid. The point of intersection of three edges is called a *vertex* of the cuboid. The three edges at any vertex are mutually perpendicular. It can be observed that a cuboid has 12 edges and 8 vertices. In figure 15.1 of a cuboid, note the following :

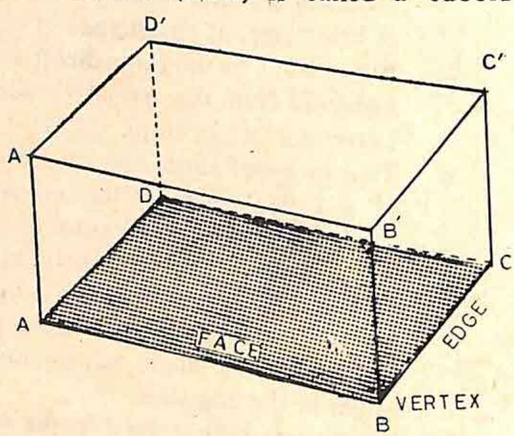



Fig. 15.1

(*i*) $A$ is one of the vertices where three *edges* $A'A$, $DA$ and $BA$ meet.

(*ii*) $B$ is a vertex where three *faces* $ABCD$, $ABB'A'$ and $BCC'B'$ meet.

(*iii*) The edges $AB$, $A'B'$, $DC$, and $D'C'$ are equal.

(*iv*) The edges $BC$, $B'C'$, $AD$ and $A'D'$ are equal.

(*v*) The edges $AA'$, $BB'$, $CC'$ and $DD'$ are equal.

We, thus, observe that the twelve edges of a cuboid can be divided into three groups as indicated in (*iii*), (*iv*) and (*v*). The measure of the edges in group (*iii*) is generally called the *length*, the measure of the edges in group (*iv*) the *breadth* or *width*, and the measure of the edges in group (*v*) the *height* or *thickness* of the cuboid.

If all the edges of a cuboid are equal, it is said to be a *cube*. So, we note that in acube, all the six faces are squares of equal area.

## TOTAL SURFACE AREA OF A CUBOID AND CUBE

If we observe a cuboid, we find that certain amount of space is enclosed inside its bounding surface which are all plane figures (rectangles). The sum of the areas of all these plane figures, forming the bounding surface is called the surface area of the cuboid.

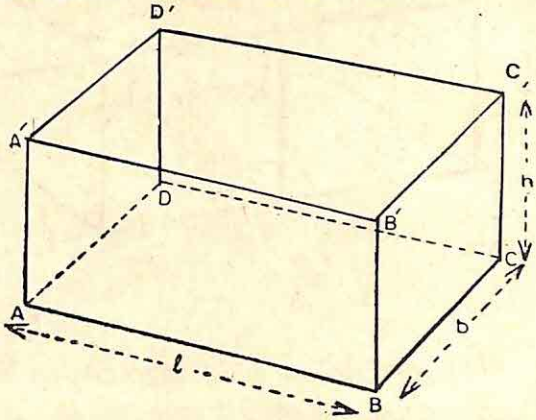



**Fig. 15.2**

Consider the cuboid ($ABCD, A'B'C'D'$) in Fig. 15.2 whose length, breadth and height are $l$, $b$ and $h$ respectively.

The surface of the cuboid consists of 3 pairs of congruent rectangles ($ABCD$, $A'B'C'D'$), ($ADD'A'$, $BCC'B'$) and ($ABB'A'$, $CDD'C'$) whose areas respectively are $(l \times b)$, $(b \times h)$ and $(h \times l)$.

Thus, the total surface area of the cuboid is $2\ (l \times b + b \times h + h \times l)$. In the case of a cube, $l = b = h$ and hence the total surface area of a cube with an edge of measure $l$ is $6\,l^2$.

## UNIT OF VOLUME

Recall that we used a unit square as the unit to measure areas of geometric figures. In the same way we choose a unit cube as unit to measure volume. A cube whose edge measures unit length is called a *unit cube*.

## VOLUME OF A CUBOID

Take 5 unit cubes. Place them one after another as in Fig. 15.3. These when so placed form a cuboid whose length is of 5 units and breadth and height are of one unit. The space occupied by the cuboid is 5 unit cubes.

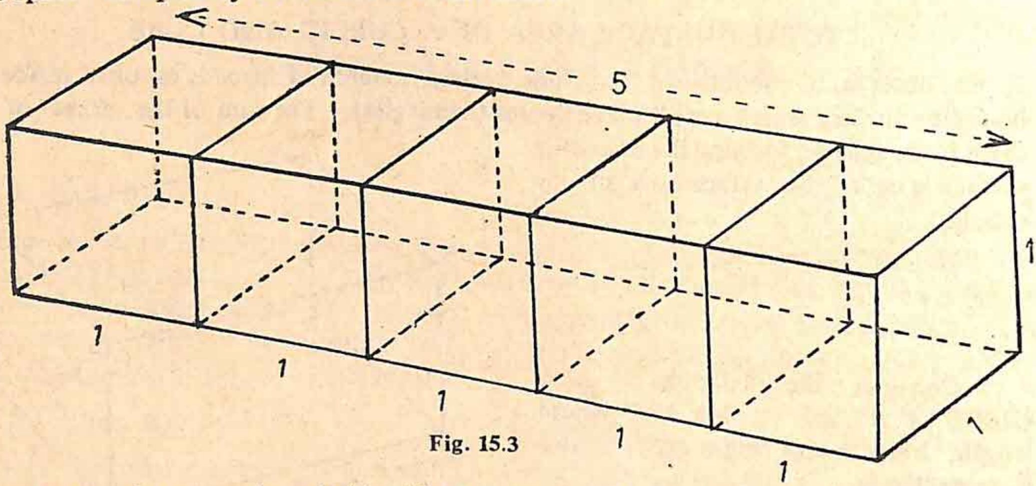



Fig. 15.3

Next, place 4 such cuboids side by side as shown in Fig. 15.4. This forms a cuboid whose length is 5 units, breadth is 4 units and height is 1 unit. It is clear that the number of unit cubes in such a cuboid is $5 \times 4$ i.e. 20.

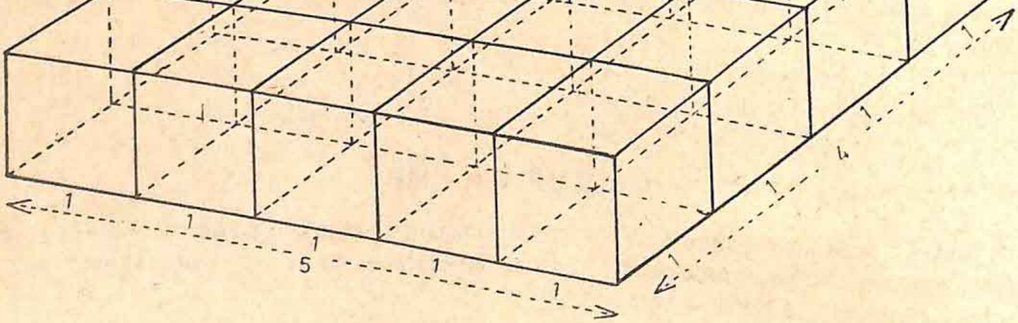



Fig. 15.4

Lastly on this cuboid of dimensions $5 \times 4 \times 1$ put two more cuboids of the same dimensions as shown in figure 15.5. Then we get a cuboid whose base is a rectangle of dimensions $5 \times 4$ and whose height is 3 units. This cuboid is, thus, made up of $5 \times 4 \times 3$ unit cubes. So a cuboid whose length, breadth and height are respectively 5, 4 and 3 units is made up of $5 \times 4 \times 3$ unit cubes.

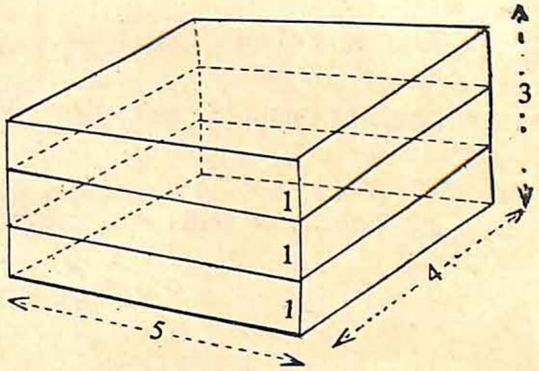


Fig 15.5

Hence, volume of this cuboid
$= 5 \times 4 \times 3$ unit cubes
$= 60$ unit cubes

From this discussion, we can arrive at a formula for the volume of a cuboid.

Volume $V$ of a cuboid is :

$$V = \text{length} \times \text{breadth} \times \text{height}$$
$$= (\text{area of the base}) \times \text{height}$$

In brief, we can write $V = l \times b \times h$, where $l$, $b$ and $h$ are length, breadth and height of the cuboid.

It follows that the volume of a cube of length $l$ is

$$V = l \times l \times l = l^3$$

The volume of a cube of edge 1 cm is 1 cubic centimetre or 1 $cm^3$ (read as one cubic centimetre).

**Example 1 :** Find the surface area and volume of a cube whose edge is 4 metres.

**Solution** : Edge $l$ of the cube $= 4$m

Surface area of the cube $= 6 \times l^2 = 6 \times (4\text{m})^2$
$= 6 \times 16 \text{ m}^2 = 96 \text{ m}^2$

Volume of the cube $= l^3 = (4 \text{ m})^3 = 64 \text{ m}^3$

**Example 2 :** The surface area of a cube is 486 sq. metres. Find its volume.

**Solution :** A cube has 6 surfaces of equal area.

$\therefore$ Area of one surface $= 486\text{m}^2 \div 6 = 81 \text{ m}^2$

$\therefore$ Edge of the cube $= \sqrt{81 \text{ m}^2} = 9 \text{ m}$

$\therefore$ Volume of the cube $= (9 \text{ m})^3 = 729 \text{ m}^3$

**Example 3 :** A tank is with rectangular base of dimensions 300 m by 150 m. With what velocity per second must water flow into it through a pipe whose rectangular cross section measures 8 dm by 5 dm so that the level may be raised $3\frac{1}{3}$ dm in 2 hours ?

**Solution** : Volume of incoming water in the tank in 2 hours is

$$\left(300 \text{ m} \times 150 \text{ m} \times \frac{1}{3} \text{ m}\right) = 15000 \text{ m}^3$$

Also, area of the cross section $= \frac{8}{10}$ m $\times$ $\frac{5}{10}$ m $= \frac{2}{5}$ m$^2$

$\therefore$ The length of water column in 2 hours $= 15000$ m$^3$ $\div \frac{2}{5}$ m$^2$

$= 37500$ m

$\therefore$ Velocity of water $= \frac{37500}{2 \times 60 \times 60}$ m/s

$= 5\frac{5}{24}$ m/s

## Exercises 15.1

1. Six cubes, each with 12 cm edge are joined end to end. Find the surface area of the resulting cuboid.
2. A box with lid is made of 2 cm thick wood. Its external length, breadth and height are 25 cm, 18 cm and 15 cm respectively. How many cubic centimetres of a liquid can be placed in it ? Find also the volume of the wood used in it.
3. An open box is made of wood 3 cm thick. Its external length is 1.48 m, breadth 1.16 m and height 8.3 dm. Find the cost of painting the inner surface at Rs 5 per m$^2$.
4. A field is 500 m long and 20 m broad and a tank 30 m long, 20 m broad and 12 m deep is dug in the field, and the earth taken out of it is spread evenly over the field. How much is the level of the field raised ?
5. The outer dimensions of a closed wooden box are 10 cm by 8 cm by 7 cm. Thickness of the wood is 1 cm. Find the total cost of wood required to make the box, if 1 cm$^3$ of wood costs Rs 2.00.
6. 500 men took dip in a tank which is 80 m long and 50 m broad. What is the rise in the water level if the average displacement of water by a man is 4 m$^3$ ?
7. Water in a canal, 30 dm wide and 12 dm deep, is flowing with velocity of 100 km an hour. How much area will it irrigate in 30 minutes if 8 cm of standing water is desired ?
8. The dimensions of a rectangular box are in the ratio of 2 : 3 : 4 and the difference between the cost of covering it with sheet of paper at the rate of Rs 4 and Rs 4.50 per square metre is Rs 416. Find the dimensions of the box.

## 15.3 Right Circular Cylinder

If we observe a lawn roller or a water pipe, we find that its two planely ends are circular and the lateral face is curved. These objects are examples of solid figures called circular cylinders. The distance between the centres of the circular faces is called the *height* of the cylinder and the line joining these centres is called the *axis* of the cylinder.

It can easily be seen that if we rotate a rectangle about one of its sides, then the solid so generated will be a right circular cylinder whose axis will be the side of the rectangle about which it is rotated and the other side will be the radius of the plane circular face of the cylinder.

### SURFACE AREA OF A CYLINDER

Consider a right circular cylinder whose height ($OO'$) is $h$ and radius ($OP$) of the base is $r$ (Fig. 15.6). The surface area of the cylinder is the sum (total) of the areas of the two plane circular faces and the area of the curved surface. In order to find the area of the curved surface of the cylinder, from the point $P$, draw a line $PQ$ perpendicular to the base, meeting the opposite face at $Q$. Suppose the cylinder is hollow and made

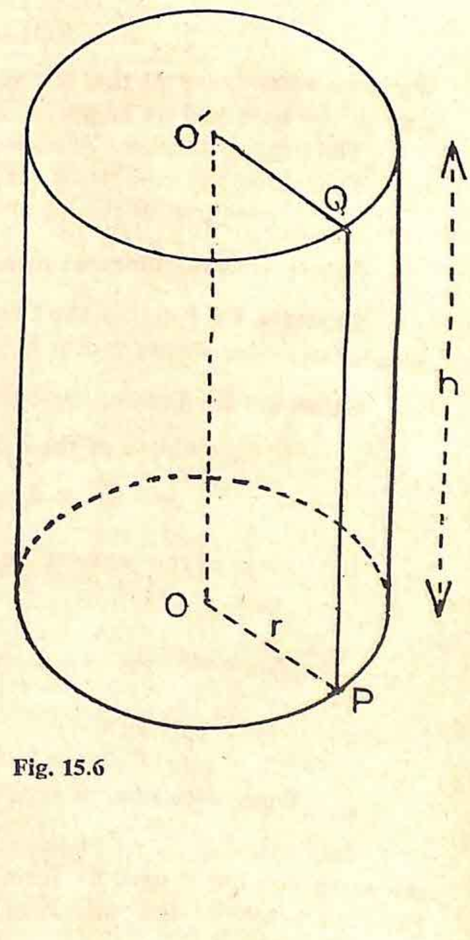



Fig. 15.6

up of card-board. If we cut the cylinder along $PQ$ and make it flat, we will obtain a rectangular sheet $PQ\,Q'P'$ (Fig. 15.7) whose width $PQ$ is equal to the height '$h$' of the cylinder and whose length $PP'$ is equal to the circumference of the base *i.e.* $2\pi r$.

Thus, the curved surface area

$$= PP' \times PQ = 2\pi rh$$

Also, area of each circular face $= \pi r^2$

$\therefore$ Total surface area $= 2\pi r^2 + 2\pi rh$

$$= 2\pi r(r + h)$$

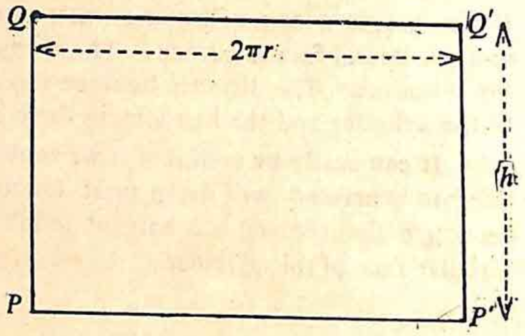


Fig. 15.7

## VOLUME OF A CYLINDER

We have already learnt that the volume of a cylinder also is given by the product of the area of the base and its height.

The area of the base of a circular cylinder of radius $r$ is $\pi r^2$.

Therefore, the volume $V$ of the cylinder of height $h$ and radius of its base $r$ =(area of the base) $\times$ height. *i.e.* $V = \pi r^2 h$.

*Note* : Unless otherwise mentioned the value of $\pi$ is taken as $\frac{22}{7}$ or 3.14.

**Example 1** : Find (*i*) the volume and (*ii*) the total surface area of a closed right circular cylinder whose radius is 5 m and height 21 m.

**Solution** : (*i*) Area of the base of the cylinder $= \pi r^2 = \frac{22}{7} \times 5\text{ m} \times 5\text{ m}$

$\therefore$ Volume of the cylinder = area of the base $\times$ height

$$= \frac{22}{7} \times 5\text{ m} \times 5\text{ m} \times 21\text{ m} = 1650\text{ m}^3$$

(*ii*) Area of the curved surface $= 2\pi rh = 2 \times \frac{22}{7} \times 5\text{ m} \times 21\text{ m}$

$$= 660\text{ m}^2$$

Area of the two ends $= 2\pi r^2 = 2 \times \frac{22}{7} \times 5\text{ m} \times 5\text{ m}$

$$= 157\tfrac{1}{7}\text{ m}^2$$

$\therefore$ Total surface area $= 660\text{ m}^2 + 157\frac{1}{7}\text{ m}^2 = 817\frac{1}{7}\text{ m}^2$

**Example 2** : A well whose internal diameter is 7 m has been sunk 22.5 m deep and the earth dug out is used to form an embankment 10.5 m wide around it (Fig. 15.8). Find the height of the embankment.

**Solution :** Radius of well $= r_1 = \frac{7}{2}$ m

Volume of the earth dug out

$$= \pi r^2 h = \frac{22}{7} \times \frac{7}{2} \text{m} \times \frac{7}{2} \text{m} \times \frac{45}{2} \text{m}$$

$$= \frac{3465}{4} \text{m}^3$$

Outer radius of embankment $= r_2$

$$= 3.5 \text{ m} + 10.5 \text{ m} = 14 \text{ m}$$

Area of the embankment

$$= \text{outer area—inner area} = \pi \left( r_2^2 - r_1^2 \right)$$

$$= \frac{22}{7}\left( (14)^2 - (3.5)^2 \right)\text{m}^2$$

$$= \frac{1155}{2}\text{m}^2$$

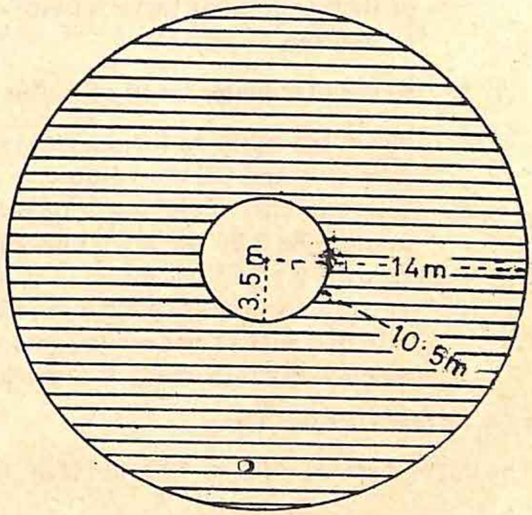


Fig. 15.8

$\therefore$ Height of the embankment

$$= \frac{\text{Volume}}{\text{Area}}$$

$$= \frac{3465}{4} \text{ m}^3 \div \frac{1155}{2} \text{ m}^2 = 1.5 \text{ m}$$

## Exercises 15.2

1. A cylindrical tower is 5 m in diameter and 14 m high. Find the cost of white-washing its curved surface at 50 paise per square metre.
2. A solid cylinder has a total surface area of 231 cm². Its curved surface area is $\frac{2}{3}$ of the total surface area. Find the volume of the cylinder.
3. Find the weight of a hollow cylindrical lead pipe 26 cm long and $\frac{1}{2}$ cm thick. Its external diameter is 5 cm. (Weight of 1 cm³ of lead is 11.4 g.)
4. The cost of painting the total outside surface of a closed cylindrical oil tank at 25 paise per square decimetre is Rs 77. The height of the tank is three times the radius of the base of the tank. Find its volume.
5. A well with inner diameter 8 m is dug 14 m deep. Earth taken out of it has been spread evenly all around it to a width of 3 m to form an embankment. Find the height of the embankment.

6. The sum of the radius of the base and the height of a solid cylinder is 37 m. If the total surface area of the solid cylinder is 1628 $m^2$, find the circumference of its base and the volume of the cylinder.
7. The height of a right circular cylinder is 6 m. Three times the sum of the areas of its two circular faces is twice the area of its curved surface. Find the radius of its base.
8. 50 circular plates, each of radius 7 cm and thickness $\frac{1}{2}$ cm, are placed one above the other to form a solid right circular cylinder. Find (*i*) the total surface area and (*ii*) the volume of the cylinder so formed.
9. Find the cost of sinking a tubewell 280 m deep having diameter 3 m at the rate of Rs 3.60 per $m^3$. Find also the cost of cementing its inner curved surface at Rs 1.25 per $m^2$.
10. Water flows out through a circular pipe whose internal diameter is 2 cm at the rate of 7 metres per second into a cylindrical tank the radius of whose base is 40 cm. By how much will the level of water rise in half an hour ?

### 15.4 Right Circular Cone

The upper portion *ABC* of a funnel (Fig. 15.9) is an example of a cone.

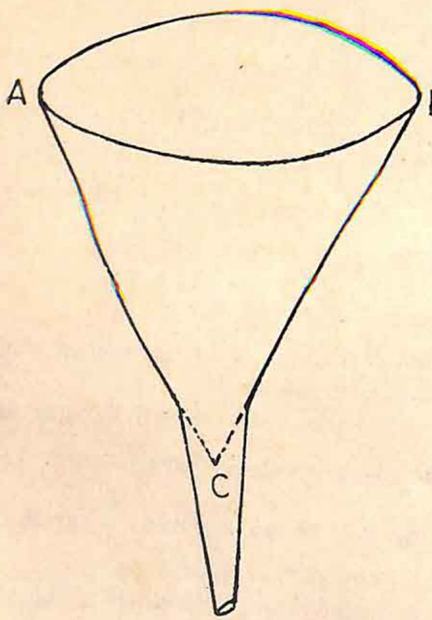



Fig. 15.9

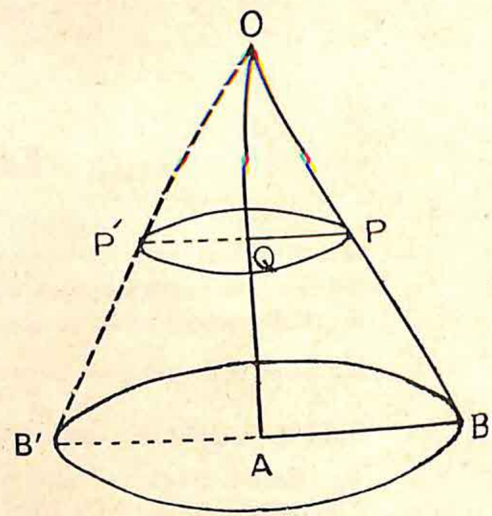



Fig. 15.10

If we take a right-angled triangular lamina *OAB* (Fig. 15.10) and rotate it along the side *OA* containing the right angle, we will find that every point *P* on the hypo-

tenuse $OB$ will generate a circle of radius $PQ$. Its length will vary from zero at $O$ to the maximum $AB$ at $A$. The solid so generated is called a *right circular cone.* It will have a curved surface and a flat surface. The flat surface will be a circle of radius $AB$. $OA$ is called the *height* of the cone, $OB$ is called its *slant height* and $O$ is called the *vertex* of the cone. If $h$ is the height of the cone and $r$ is the radius of the base of the cone, we obtain from Pythagoras theorem, the slant height $OB$ given by

$$OB^2 = OA^2 + AB^2$$

$$\therefore OB = \sqrt{h^2 + r^2}$$

## CURVED SURFACE AREA OF A RIGHT CIRCULAR CONE

Consider a hollow cone $OBB'$ made of paper. Let its height be $h$, radius of the base $AB = r$, and slant height $OB = l$. Let us cut the cone along the line $OB$ and open it as shown in Fig. 15.11. Evidently the shape of the curved surface will be the sector of a circle whose centre is $O$ and radius equal to $OB$ and length of the arc $BCB'$ equal to the circumference of the circular base of the cone.

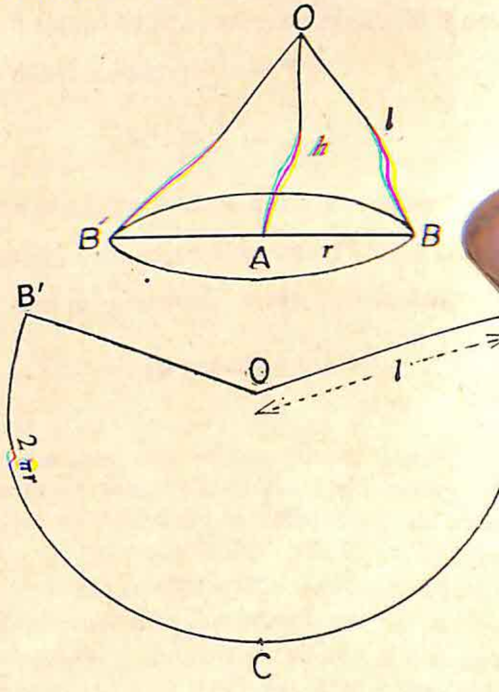



Fig. 15.11

$$\frac{\text{Area of the sector } OBCB'}{\text{Area of the circle with centre } O \text{ and radius } l}$$

$$= \frac{\text{Arc } BCB'}{\text{Circumference of a circle with centre } O \text{ and radius } l}$$

$$= \frac{2\pi r}{2\pi l} = \frac{r}{l}$$

or, area of the sector $OBCB' = \frac{r}{l}$ (area of the circle with centre $O$ and radius $l$)

$$= \frac{r}{l} \times \pi l^2 = \pi r l$$

Thus, area of the curved surface of a cone $= \pi r l$

Hence, total surface area of the cone

area of the curved surface + area of the circular base $= \pi r l + \pi r^2 = \pi r (l + r)$

## VOLUME OF A CONE

The volume of a right circular cone is one-third of the volume of a right circular cylinder of the same base and same height (the proof is beyond the scope of this book). Hence, the volume $V$ of a right circular cone of height $h$ and radius $r$ is given by

$$V = \frac{1}{3} \text{ (volume of the cylinder of height } h \text{ and radius } r)$$

$$= \frac{1}{3} \pi r^2 h$$

**Example 1** : The vertical height of a conical tent is $4\frac{2}{3}$ m and the radius of its base is 6 m. It can accommodate 11 persons. Find the average air space per person.

**Solution** : Here $h = \frac{14}{3}$ m and $r = 6$ m.

$\therefore$ Volume enclosed by the tent $= \frac{1}{3} \pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times 6\text{m} \times 6\text{m} \times \frac{14}{3}\text{m}$

$$= 176 \text{ m}^3$$

Average air space per person $= (176 \div 11) \text{ m}^3 = 16 \text{ m}^3$

**Example 2** : A bucket made up of tin sheet is of height 20 cm and has its upper and lower ends of radius 24 cm and 8 cm, respectively (Fig. 15.12). Find the cost of bucket full of milk, at the rate of Rs 5 per litre and the cost of the tin sheet used if the cost of tin sheet is Re 1.00 per 100 cm$^2$.
(1000 cm$^3$ = 1 litre)

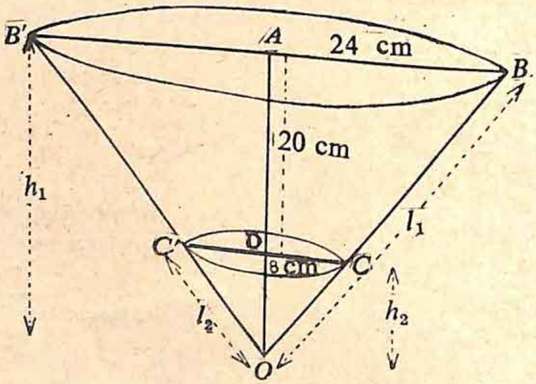


Fig. 15.12

**Solution** : The volume of bucket is the difference of volumes of two cones $OBB'$ and $OCC'$ of heights $h_1$ and $h_2$ respectively.

From $\triangle$s $OBA$ and $OCD$, we have

$$\frac{24}{8} = \frac{h_1}{h_2} \Rightarrow h_1 = 3\,h_2$$

But $\qquad h_1 = 20 + h_2$

$\therefore \qquad 3h_2 = 20 + h_2$

$\therefore \qquad h_2 = 10$ cm

$\therefore \qquad h_1 = 20 \text{ cm} + 10 \text{ cm} = 30 \text{ cm}$

Volume of the bucket $= \frac{1}{3}\ \pi\ (24\ \text{cm})^2 \times 30\ \text{cm} - \frac{1}{3}\ \pi\ (8\ \text{cm})^2 \times 10\ \text{cm}$

$$= \frac{22 \times 640 \times 26}{21}\ \text{cm}^3$$

Cost of milk $= \text{Rs}\left(\frac{22 \times 640 \times 26}{21} \times \frac{5}{1000}\right) = \text{Rs } 87.16$

(*ii*) Let $l_2$ and $l_1$, be the slant heights of the cones $OCC'$ and $OBB'$ respectively.

Then, $\qquad l_2 = \sqrt{8^2 + 10^2}$ cm

$= 12.8$ cm

and $\qquad l_1 = \sqrt{24^2 + 30^2}$ cm

$= 38.4$ cm

Surface area of the bucket $= \pi r_1 l_1 - \pi\, r_2 l_2 + \pi\, r_2^2$

$$= \frac{22}{7} \times 24\ \text{cm} \times 38.4\ \text{cm} - \frac{22}{7} \times 8\ \text{cm} \times 12.8\ \text{cm} + \frac{22}{7} \times 8\ \text{cm} \times 8\ \text{cm}$$

$$= 2775.8\ \text{cm}^2$$

Cost of the bucket $= \text{Rs } 2775.8 \times \frac{1}{100} = \text{Rs } 27.76$

## Exercises 15.3

1. Find the slant height of a cone whose volume is 1232 cm$^3$ and the radius of the base 7 cm.
2. If the diameter of a cone is 14 cm and its slant height 9 cm, find the area of its curved surface.
3. The slant height and the diameter of a conical tomb are 25m and 14m respectively. Find the cost of constructing it at Rs 2 per m$^3$ and also cost of white-washing its curved surface at 80 paise per 100m$^2$.
4. The material of a solid cone is converted into the shape of a solid cylinder of equal radius. If the height of the cylinder is 5 cm, what is the height of the cone ?
5. The radius and the height of a right circular cone are in the ratio of 5 : 12. If its volume is 314 cm$^3$, find the slant height and the radius of the cone (use $\pi = 3.14$).

6. A conical vessel whose internal dimensions are 105 cm deep and 120 cm in diameter is full of liquid. If a cubic decimetre of liquid weighs 1 kg 500 g, find the weight of liquid contained in the vessel.
7. How many metres of cloth 3 m wide will be required to make a conical tent whose base radius is 12m and height is 16m ?
8. A circus tent consists of a cylindrical base surmounted by a conical roof. The radius of the cylinder is 20m. The heights of the cylindrical and conical portions are respectively 42m and 21m. Find the volume of air contained in he tent.
9. The height of a cone is 30 cm. A small cone is cut off at the top by a plane parallel to the base. If its volume be $\frac{1}{27}$ of the volume of the given cone, at what height above the base is the section made ?
10. A tent of height 33m is in the form of a right circular cylinder of diameter 120m and height 22m surmounted by a right circular cone of the same diameter. Find the cost of canvas of the tent at the rate of Rs 1.50 per $m^2$.

### 15.5 Sphere

If we take a semi-circle *AOBCA* (Fig. 15.13) and rotate it about its diameter *AB*, the solid so generated by a complete revolution of the semi-circle about the diameter *AB* is called a *sphere*. A ball is an example of a sphere.

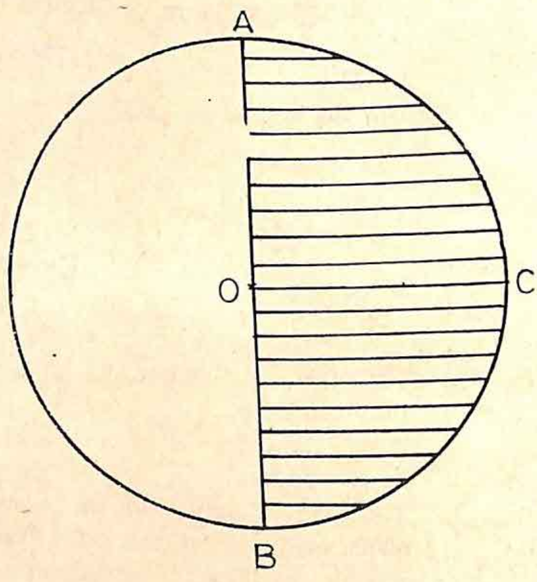



Fig. 15.13

The centre $O$ and the radius $OA$ or $OB$ of the semi-circle are also called the *centre* and *radius of the sphere*.

It will be observed that if we cut the sphere by any plane, then the section of the sphere will be a circle.

The radius of the circular section will decrease as we move the plane away from

the centre of the sphere. In fact, the section that passes through the centre of the sphere is the largest, called a *great circle*. Such a plane cuts the sphere in two equal parts, each called a *hemisphere*.

## VOLUME AND SURFACE AREA OF A SPHERE

We state without proof (which is beyond the scope of the present book) that the surface area $A$ (say) and volume $V$ of a sphere of radius $r$ are given by

$$A = 4\pi r^2$$

$$V = \frac{4}{3}\pi r^3$$

**Example 1** : Find ($i$) the surface area and ($ii$) the volume of a sphere whose diameter is 14 cm.

**Solution** : Diameter = 14 cm

$\therefore$ radius ($r$) = 7 cm

($i$) Surface area of the sphere $= 4\pi r^2$

$$= 4 \times \frac{22}{7} \times 7 \text{ cm} \times 7 \text{ cm} = 616 \text{ cm}^2$$

($ii$) Volume of the sphere $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 7 \text{ cm} \times 7 \text{ cm} \times 7 \text{ cm}$$

$$= 1437\frac{1}{3} \text{ cm}^3$$

**Example 2** : Three solid metallic spheres of radii 6 cm, 8 cm and 10 cm respectively are melted to form a single solid sphere. Find the radius of the resulting sphere.

**Solution** : The volumes of the spheres of radii 6 cm, 8 cm and 10 cm are respectively $\frac{4}{3}\pi \times 6^3 \text{ cm}^3$, $\frac{4}{3}\pi \times 8^3 \text{ cm}^3$ and $\frac{4}{3}\pi \times 10^3 \text{ cm}^3$.

If $r$ cm be the radius of the resulting sphere, then

$$\frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 6^3 + \frac{4}{3}\pi \times 8^3 + \frac{4}{3}\pi \times 10^3$$

Dividing both sides by $\frac{4}{3}\pi$, we get

$$r^3 = 6^3 + 8^3 + 10^3 = 1728 = 12^3$$

$$\therefore \quad r = 12$$

*i.e.* Radius of the resulting sphere = 12 cm.

## Exercises 15.4

1. How many lead shots each 0.3 cm in diameter can be made from a cuboid of dimensions 9 cm $\times$ 11 cm $\times$ 12 cm ?
2. A sphere has the same curved surface as the curved surface of a cone of height 360 cm and base radius 150 cm. Find the radius of the sphere.
3. A sphere of diameter 5 cm is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 10 cm. If the sphere is completely submerged, by how much will the surface of water rise ?
4. The radii of the internal and external surfaces of a hollow metallic sphere are 3 cm and 5 cm respectively. It is melted and its material cast into a solid cylinder of height 10 $\frac{2}{3}$ cm. Find the diameter of the cylinder.
5. A toy is in the form of a cone mounted on a hemisphere. The diameter of the base of the cone is 6 cm and its height 4 cm. Calculate the surface area of the toy. (use $\pi = 3.14$)
6. A spherical shell of lead whose external diameter is 18 cm is melted and recast into a right circular cylinder 8 cm high and 12 cm in diameter. Find the inner diameter of the shell.
7. How many lead balls each of radius 1 cm can be made from a sphere whose radius is 8 cm ?
8. The diameter of a sphere is 6 cm. It is melted and drawn into a wire of diameter 0.2 cm. Find the length of the wire.
9. A spherical ball of lead 3 cm in diameter is melted and recast into 3 spherical balls. The diameter of the two of these are 1 cm and 1.5 cm respectively. What is the diameter of the third spherical ball ?

CHAPTER 16

# Arithmetical Descriptors of Statistical Data

## 16.1 Measures of Central Tendency

In Class IX, statistical data were represented graphically to get some broad general idea about the nature of the data. The most important objective of statistical analysis is to obtain representative measures each of which describes a characteristic of the entire data. An average is one of such representative measures. According to Professor Bowley, "averages are statistical constants which enable us to comprehend in a single effort the significance of the whole". They generally give us an idea about the concentration of the values in the central part of the distribution. In fact, an average of a statistical data is the value which is representative of the entire data. As average represents the entire data, its value lies somewhere in between two extremes. For this reason, average is called a measure of central tendency. Normally most of the values cluster around this average. It is in this sense that arithmetic average is also called *arithmetic descriptor of location.*

### REQUISITES FOR AN IDEAL MEASURE OF LOCATION

According to Professor Yule, an ideal measure of central tendency (location) should possess the following characteristics :

(*a*) It should be rigidly defined.

(*b*) It should be easily understandable and easy to calculate.

(*c*) It should be based on all the observations.

(*d*) It should be suitable for further algebraic treatment, i.e. if we are given the averages and sizes of a number of distributions, we should be able to calculate the average of the composite distribution obtained on combining the given distributions.

(*e*) It should be affected as little as possible by fluctuations of sampling.

The measures of central tendency that are in common use are :

(*i*) Arithmetic mean or mean

(*ii*) Median

(*iii*) Mode

(*iv*) Geometric mean

(*v*) Harmonic mean

In this chapter, we study only mean, median and mode.

## 16.2 Mean (Arithmetic Mean)

### MEAN OF RAW DATA

To calculate the mean of raw data, the observations are added and their sum is divided by the number of observations. Thus, the arithmetic mean (i.e. mean) of $n$ observations $x_1, x_2, x_3, \ldots, x_n$ is

$$\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} \text{ and is denoted by } \bar{x}.$$

$$\therefore \quad \bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

The symbol "$\sum$" is the capital letter 'capital sigma' of the Greek alphabet and is used in mathematics to denote summation.

**Example 1 :** In an examination, 10 students obtained 5, 7, 8, 11, 13, 9, 12, 14, 15, 16 marks out of 20 marks. Find the mean.

**Solution :** Number of students $(n) = 10$

$$\text{Sum of marks obtained } \left(\sum_{i=1}^{10} x_i\right) = 5+7+8+11+13+9+12+14+15+16$$
$$= 110$$

$$\therefore \quad \text{Mean } (\bar{x}) = \frac{1}{n}\sum x_i = \frac{110}{10} = 11 \text{ marks}$$

**Example 2 :** Maximum temperatures in Celcius for 7 days of a week in the month of May are as follows :

40.1°, 39.3°, 39.5°, 39.7°, 40°, 40.2°, 39.1°

Find the mean maximum temperature.

**Solution :**

$$\text{Mean} = \frac{\sum_{i=1}^{7} x_i}{7} = \frac{40.1 + 39.3 + 39.5 + 39.7 + 40 + 40.2 + 39.1}{7}$$

$$= \frac{277.9}{7} = 39.7^\circ \text{ Celcius}$$

## MEAN OF UNGROUPED DATA

In case of a discrete frequency distribution $(x_i, f_i)$, $i = 1, 2, \ldots, n$ where $f_i$ is the frequency of the variable value $x_i$, variable values are multiplied by their frequencies and their sum (i.e. sum of the products) is divided by the sum of the frequencies to obtain the mean. That is :

$$\text{Mean } \bar{x} = \frac{f_1 x_1 + f_2 x_2 + \ldots + f_n x_n}{f_1 + f_2 + \ldots + f_n} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

**Formula 1 :** $\bar{x} = \dfrac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$

**Example 3 :** The following data represents the earnings of 50 labourers in a factory:

| Earnings (in rupees) | 450 | 475 | 500 | 525 | 550 |
|---|---|---|---|---|---|
| Number of labourers | 12 | 13 | 7 | 10 | 8 |

Calculate the average earnings of a labourer.

**Solution :**

| *Earnings* $(x)$ | *Number of labourers* $(f)$ | $f\,x$ |
|---|---|---|
| 450 | 12 | 5400 |
| 475 | 13 | 6175 |
| 500 | 7 | 3500 |
| 525 | 10 | 5250 |
| 550 | 8 | 4400 |
| | $\sum_{i=1}^{5} f_i = 50$ | $\sum_{i=1}^{5} f_i x_i = 24725$ |

$$\text{Average earnings} = \frac{\sum_{i=1}^{5} f_i x_i}{\sum_{i=1}^{5} f_i} = \text{Rs}\,\frac{24725}{50}$$

$$= \text{Rs}\ 494.50$$

## SHORT CUT METHOD FOR FINDING MEAN OF UNGROUPED DATA

It may be noted that if the values of $x$ and/or $f$ are large, calculation of the mean by formula 1 is quite time consuming and tedious. The computations are reduced to a great extent by taking the deviations of the given values from any arbitrary point '$a$' (called assumed mean), as explained below:

Let $d_i$ = deviation of $x_i$ from '$a$'

$$= x_i - a$$

$$\therefore \quad \sum_{i=1}^{n} f_i d_i = \sum_{i=1}^{n} f_i (x_i - a)$$

$$\therefore \quad \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i (x_i - a)}{\sum_{i=1}^{n} f_i}$$

$$= \frac{\sum_{i=1}^{n} f_i x_i - \sum_{i=1}^{n} a f_i}{\sum_{i=1}^{n} f_i}$$

$$= \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} - \frac{a \sum_{i=1}^{n} f}{\sum_{i=1}^{n} f_i}$$

$$= \bar{x} - a$$

$$\therefore \qquad \bar{x} = a + \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}$$

i.e. $$\bar{x} = a + \bar{d},$$

where $$\bar{d} = \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}$$

**Formula 2 :** $\bar{x} = a + \bar{d}$, where $\bar{d} = \dfrac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}$

It may be noted that $\bar{d}$ is the mean of deviations.

*Note* : Any number can serve the purpose of the arbitrary point '$a$' but, usually, a value of $x$ somewhere in the middle part of the distribution will be much more convenient.

**Example 4 :** Using a convenient assumed mean (the short cut method), find the mean of the following frequency distribution :

| $x$ | 50 | 53 | 57 | 61 | 63 | 65 | 69 | 71 | 74 | 79 |
|---|---|---|---|---|---|---|---|---|---|---|
| $f$ | 31 | 35 | 47 | 25 | 26 | 48 | 51 | 63 | 41 | 33 |

**Solution :** Let assumed mean ($a$) be 65.

| $x$ | $f$ | $d = (x - a)$ | $fd$ |
|---|---|---|---|
| 50 | 31 | $-15$ | $-465$ |
| 53 | 35 | $-12$ | $-420$ |
| 57 | 47 | $-8$ | $-376$ |
| 61 | 25 | $-4$ | $-100$ |
| 63 | 26 | $-2$ | $-52$ |
| 65 | 48 | 0 | 0 |
| 69 | 51 | 4 | 204 |
| 71 | 63 | 6 | 378 |
| 74 | 41 | 9 | 369 |
| 79 | 33 | 14 | 462 |
| | 400 | | $-1413 + 1413 = 0$ |

Using formula 2, we have

$$\bar{x} = a + \bar{d}$$

$$= a + \frac{\sum_{i=1}^{10} f_i d_i}{\sum_{i=1}^{10} f_i}$$

$$= 65 + \frac{0}{400}$$

$$= 65$$

*Note* : Had the problem been solved by the first method, tedious calculations like $50 \times 31$, $53 \times 35$, $57 \times 47$, $61 \times 25$, $63 \times 26$, $65 \times 48$, $69 \times 51$, $71 \times 63$, $74 \times 41$, $79 \times 33$ would have to be done. That is why short cut method is preferred in such questions.

## PROPERTIES OF ARITHMETIC MEAN

**Property 1 :** Algebraic sum of the deviations of a set of values from their arithmetic mean is zero. That is, if $(x_i, f_i)$, $i = 1, 2, \ldots, n$ is the frequency distribution, then $\sum_{i=1}^{n} f_i (x_i - \bar{x}) = 0$, $\bar{x}$ being the mean of the distribution.

*Proof* :
$$\sum_{i=1}^{n} f_i\,(x_i - \bar{x}) = \sum_{i=1}^{n} (f_i x_i - f_i \bar{x})$$
$$= \sum_{i=1}^{n} f_i x_i - \sum_{i=1}^{n} f_i \bar{x}$$
$$= \frac{\left(\sum_{i=1}^{n} f_i\right)\left(\sum_{n=1}^{n} f_i x_i\right)}{\sum_{i=1}^{n} f_i} - \bar{x} \sum_{i=1}^{n} f_i$$
$$= \left(\sum_{i=1}^{n} f_i\right)\left(\bar{x}\right) - \bar{x} \sum_{i=1}^{n} f_i$$
$$= 0$$

**Property 2** : If $\bar{x}_i$ $(i = 1, 2, \ldots, p)$ are means of $p$ frequency distributions with respective total frequencies $n_1, n_2, \ldots, n_p$, then the mean $\bar{x}$ of the composite distribution obtained on combining the component distributions is given by formula

$$\bar{x} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2 + \ldots + n_p \bar{x}_p}{n_1 + n_2 + \ldots + n_p}$$

We justify the truthfulness of this property with the help of the following example :

**Example 5** : The mean of the marks secured by 25 students of Section A of Class X is 47, that of 35 students of Section B is 51 and that of 30 students of Section C is 53. Find the mean of the marks secured by 90 students of Class X.

**Solution** : Mean of marks of 25 $(= n_1)$ students of Section A $= 47\ (= \bar{x}_1)$

$\therefore$ Total marks obtained by 25 students of Section A
$= n_1 \bar{x}_1 = 25 \times 47 = 1175$

Mean of marks of 35 $(= n_2)$ students of Section B $= 51\ (= \bar{x}_2)$

$\therefore$ Total marks obtained by 35 students of Section B
$= n_2 \bar{x}_2 = 35 \times 51 = 1785$

Mean of marks of 30 $(= n_3)$ students of Section C $= 53\ (= \bar{x}_3)$

$\therefore$ Total marks obtained by 30 students of Section C
$= n_3 \bar{x}_3 = 30 \times 53 = 1590$

$\therefore$ Total marks obtained by 90 students of Class X
$= n_1 \bar{x}_1 + n_2 \bar{x}_2 + n_3 \bar{x}_3 = 1175 + 1785 + 1590$
$= 4550$

$\therefore$ Mean of marks obtained by 90 ($= n_1 + n_2 + n_3$) students

$$= \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2 + n_3 \bar{x}_3}{n_1 + n_2 + n_3}$$

$$= \frac{4550}{90}$$

$$= 50.56 \text{ (approximately)}$$

**Example 6 :** The average score of girls in Class X examination in a school is 73 and that of boys is 71. The average score in Class X examination in that school is 71.8. Find the percentage of girls and boys in Class X of the school.

**Solution :** Let $n_1$ be the number of girls and $n_2$ be the number of boys.

Average score of girls ($= \bar{x}_1$) $= 73$

Average score of boys ($= \bar{x}_2$) $= 71$

Average score of students of Class X ($= \bar{x}$) $= 71.8$

We know that $\bar{x} = \dfrac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$

$$\therefore \quad 71.8 = \frac{n_1 \times 73 + n_2 \times 71}{n_1 + n_2}$$

$$\therefore \quad (71.8)(n_1 + n_2) = 73n_1 + 71n_2$$

*i.e.* $\quad .8n_2 = 1.2n_1$

*i.e.* $\quad \dfrac{n_1}{n_2} = \dfrac{2}{3}$

$\therefore$ Percentage of girls in Class X $= \dfrac{2}{5} \times 100 = 40\%$

and percentage of boys in Class X $= \dfrac{3}{5} \times 100 = 60\%$

## Exercises 16.1

1. The marks obtained by 10 students are :

   22, 35, 37, 38, 29, 27, 34, 36, 28, 34

   Find the mean of marks.

2. Find the mean of the following frequency distribution :

| $x$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| $f$ | 12 | 11 | 12 | 13 | 9 | 10 | 14 | 7 |

3. From the following table, find the mean of the weekly wages :

| Weekly wages (in rupees) | 45 | 50 | 55 | 60 | 65 | 70 | 75 |
|---|---|---|---|---|---|---|---|
| Number of employees | 12 | 13 | 14 | 13 | 12 | 11 | 5 |

4. Find the mean of the following marks obtained by students of a Class:

| Marks | 15 | 20 | 25 | 30 | 35 | 40 |
|---|---|---|---|---|---|---|
| Number of students | 6 | 7 | 12 | 14 | 15 | 6 |

5. The mean height of 10 students is 153 cm. It is discovered later on that while calculating the mean the reading 151 cm was wrongly read as 141 cm. Find the correct mean.
6. The mean monthly wage of a group of 11 persons is Rs 1450. One member of the group, whose monthly wage is Rs 1500, left the group. Find the average monthly wage of the remaining 10 members of the group.
7. The mean of weights of the 25 students of Section A of a Class is 51 kg whereas the mean of weights of 35 students of Section B of the same Class is 54 kg. Find the mean of the weights of all the 60 students of that Class.
8. The mean monthly salary paid to 75 employees in a company is Rs 1420. The mean salary of 25 of them is Rs 1350 and that of 30 others is Rs 1425. What is the mean salary of the remaining ?
9. The mean of 11 numbers is 23. If 5 is added in every number, find the new mean.
10. The mean of 121 numbers is 59. If each number is multiplied by 4, what will be the new mean ?

## MEAN OF GROUPED DATA

In case of grouped data, the classes are replaced by their representative values which are the mid-values of the classes. Taking these mid-values as the variable values, the mean of the grouped data is calculated as in the case of the ungrouped data.

The assumption that the frequency of a class is centred at the mid-point of the class may lead to loss of accuracy. Suppose, for example, we have four scores 6, 7, 8 and 9 in a class 0-10. The total of these is 30. When we multiply the mid-value 5 of the class by its frequency 4, we get 20. Thus, we are losing 10 from the actual total score of the class and so accuracy is lost. Usually the scores are scattered in the class and the assumption that its frequency is centred at mid-point of the class leads to the

inaccuracy of the type as explained above. Anyhow, the inaccuracy is normally not very large and therefore, the result obtained on the basis of the above assumption is approximately correct.

If raw data is given and it is required to find their mean, we do not change the raw data to grouped data unless it is specifically asked. Converting raw data into grouped data makes the problem of finding mean computationally simpler but at the cost of accuracy. Mean of the ungrouped data is the most accurate one.

Some examples are worked out below to illustrate the procedure:

**Example 1 :** Find the average income from the following frequency distribution :

| *Income Group (in rupees)* | *Number of workers* |
|---|---|
| 200— 300 | 13 |
| 300— 400 | 11 |
| 400— 500 | 15 |
| 500— 600 | 16 |
| 600— 700 | 18 |
| 700— 800 | 13 |
| 800— 900 | 14 |
| 900—1000 | 10 |
| | 110 |

**Solution :** Let the assumed mean '$a$' be 600.

| *Class* | *Mid-value of class (x)* | $d = x - 600$ | *Frequency f* | *fd* |
|---|---|---|---|---|
| 200— 300 | 250 | − 350 | 13 | − 4550 |
| 300— 400 | 350 | − 250 | 11 | − 2750 |
| 400— 500 | 450 | − 150 | 15 | − 2250 |
| 500— 600 | 550 | − 50 | 16 | − 800 |
| 600— 700 | 650 | 50 | 18 | + 900 |
| 700— 800 | 750 | 150 | 13 | + 1950 |
| 800— 900 | 850 | 250 | 14 | + 3500 |
| 900—1000 | 950 | 350 | 10 | + 3500 |
| | | | $\sum f_i = 110$ | $\sum f_i d_i = -500$ |

$\therefore \quad \bar{d} = -\dfrac{500}{110} = -4.55$ (approximately)

$\therefore$ By formula 2, $\bar{x} = 600 + \bar{d} = 600 - 4.55$

$\therefore$ Average income = Rs 595.45

In the case of grouped frequency distribution with equal class widths, the computation of arithmetic mean is further simplified by taking

$$d_i = \frac{x_i - a}{h} \quad i.e., \; x_i = a + hd_i,$$

where '$a$' is the arbitrary point (called assumed mean), $h$ is the common magnitude of class intervals and $x_i$ is the mid-point of the $i^{th}$ class. In this case, we have

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

$$= \frac{\sum_{i=1}^{n} f_i (a + hd_i)}{\sum_{i=1}^{n} f_i}$$

$$= \frac{a \sum_{i=1}^{n} f_i + h \sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}$$

$$= a + h\bar{d}$$

**Formula 3 :** $\bar{x} = a + h\bar{d}$

**Example 2 :** For the following frequency table, find the mean :

| *Class* | *Frequency* |
|---|---|
| 20— 40 | 9 |
| 40— 60 | 11 |
| 60— 80 | 14 |
| 80—100 | 6 |
| 100—120 | 8 |
| 120—140 | 15 |
| 140—160 | 10 |
| 160—180 | 20 |
| 180—200 | 7 |
| | 100 |

**Solution :** Let the assumed mean ($a$) be 110. In the present case, common width ($h$) of the classes is 20.

| *Class* | *Mid-value* ($x$) | *Frequency* ($f$) | $d = \frac{x - 110}{20}$ | $fd$ |
|---|---|---|---|---|
| 20—40 | 30 | 9 | —4 | —36 |
| 40—60 | 50 | 11 | —3 | —33 |
| 60—80 | 70 | 14 | —2 | —28 |
| 80—100 | 90 | 6 | —1 | —6 |
| 100—120 | 110 | 8 | 0 | 0 |
| 120—140 | 130 | 15 | 1 | 15 |
| 140—160 | 150 | 10 | 2 | 20 |
| 160—180 | 170 | 20 | 3 | 60 |
| 180—200 | 190 | 7 | 4 | 28 |
| | | 100 | | 20 |

$$\therefore \qquad \bar{d} = \frac{20}{100} = \frac{1}{5}$$

By formula 3,

$$\bar{x} = a + h\,\bar{d}$$

$$= 110 + 20 \times \frac{1}{5} = 114$$

**Example 3 :** Find the arithmetic mean of the following frequency distribution:

| *Marks* | *Number of students* |
|---|---|
| Below 10 | 12 |
| Below 20 | 22 |
| Below 30 | 35 |
| Below 40 | 50 |
| Below 50 | 70 |
| Below 60 | 86 |
| Below 70 | 97 |
| Below 80 | 104 |
| Below 90 | 109 |
| Below 100 | 115 |

**Solution** : Rewriting the distribution in the form of grouped distribution with each class interval as 10 and taking assumed mean to be 55, we get the following table:

| *Class* | *Mid-value* ($x$) | *Number of students* ($f$) | $d = \frac{x-55}{10}$ | $fd$ |
|---|---|---|---|---|
| 0—10 | 5 | 12 | −5 | −60 |
| 10—20 | 15 | 10 | −4 | −40 |
| 20—30 | 25 | 13 | −3 | −39 |
| 30—40 | 35 | 15 | −2 | −30 |
| 40—50 | 45 | 20 | −1 | −20 |
| 50—60 | 55 | 16 | 0 | 0 |
| 60—70 | 65 | 11 | +1 | 11 |
| 70—80 | 75 | 7 | +2 | 14 |
| 80—90 | 85 | 5 | +3 | 15 |
| 90—100 | 95 | 6 | +4 | 24 |
| | | 115 | | −125 |

$$\bar{d} = \frac{\sum_{i=1}^{10} f_i d_i}{\sum_{i=1}^{10} f_i} = -\frac{125}{115} = -\frac{25}{23}$$

By formula 3,

$$\begin{aligned} \bar{x} &= 55 + 10 \times \left(-\frac{25}{23}\right) \\ &= 55 - \frac{250}{23} \\ &= 55 - 10.87 \text{ (approximately)} \\ &= 44.13 \text{ (approximately)} \end{aligned}$$

$\therefore$ Mean $= 44.13$ (approximately)

## WEIGHTED ARITHMETIC MEAN

Arithmetic mean attaches equal importance to all the observations, but there are situations when same importance is not attached to all the observations. For example, for admission to Medical Courses more importance is attached to Physics, Chemistry and Biology as compared to languages. Similarly for admission to I.I.T.'s more weightage is attached to Mathematics, Physics and Chemistry

If $x_i$ $(i = 1, 2, \ldots, n)$ are $n$ observations to which weights attached are $w_i$ $(i = 1, 2, \ldots, n)$, then weighted mean is defined as:

$$\textbf{Weighted mean } (\bar{x}_w) = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i}$$

**Example 4** : The following table gives the marks of a student in Class XII examination and weights attached to each subject by the selection board of a professional college:

| *Subject* | *Weight* | *Marks* |
|---|---|---|
| English | 1 | 62 |
| Mathematics | 3 | 83 |
| Physics | 3 | 79 |
| Chemistry | 2 | 74 |
| Engineering Drawing | 2 | 77 |

Compare the mean with the weighted mean.

**Solution** : $\text{Mean} = \frac{62 + 83 + 79 + 74 + 77}{5}$

$$= \frac{375}{5} = 75 \text{ marks}$$

$$\text{Weighted mean} = \frac{1 \times 62 + 3 \times 83 + 3 \times 79 + 2 \times 74 + 2 \times 77}{11}$$

$$= \frac{62 + 249 + 237 + 148 + 154}{11}$$

$$= \frac{850}{11}$$

$$= 77.3 \text{ marks (approximately)}$$

$\therefore$ Weighted mean is more than the mean.

**Remarks** :

(*a*) If weight of each observation is same, weighted mean is same as mean.

(*b*) If weights attached to higher values are greater than those attached to smaller values, then weighted mean becomes more than the mean (as seen in Example 4).

(*c*) If weights attached to smaller values are greater than those attached to higher values, then weighted mean becomes less than the mean.

For instance, if in *Example 4* weight attached to English is 4, to Mathematics is 1, to Physics is 1, to Chemistry is 2 and to Engineering Drawing is 1, then weighted mean would be = 70.6 marks (approx.) which is less than the mean.

## Exercises 16.2

1. Compute the mean of the following distribution of marks obtained by 125 students in statistics in an examination:

| *Marks* | *Number of students* |
|---|---|
| 0—10 | 3 |
| 10—20 | 4 |
| 20—30 | 8 |
| 30—40 | 10 |
| 40—50 | 15 |
| 50—60 | 35 |
| 60—70 | 20 |
| 70—80 | 16 |
| 80—90 | 8 |
| 90—100 | 6 |

2. For the following frequency table, find the mean :

| *Monthly rent (in rupees)* | *Number of families paying rent* |
|---|---|
| 800—1000 | 43 |
| 1000—1200 | 57 |
| 1200—1400 | 63 |
| 1400—1600 | 52 |
| 1600—1800 | 45 |
| 1800—2000 | 37 |
| 2000—2200 | 23 |

3. Find the mean of the following distribution :

| *Marks* | *Frequency* |
|---|---|
| More than or equal to 80 | 0 |
| More than or equal to 75 | 4 |
| More than or equal to 70 | 11 |
| More than or equal to 65 | 22 |
| More than or equal to 60 | 37 |
| More than or equal to 55 | 45 |
| More than or equal to 50 | 50 |

4. Find the average income from the following data :

| *Income (in rupees)* | *Number of workers* |
|---|---|
| More than or equal to 500 | 127 |
| More than or equal to 600 | 120 |
| More than or equal to 700 | 112 |
| More than or equal to 800 | 105 |
| More than or equal to 900 | 101 |
| More than or equal to 1000 | 95 |
| More than or equal to 1100 | 70 |
| More than or equal to 1200 | 57 |

5. Following are the data of the daily wages of workers employed by an organisation. Compute the average wage.

| *Daily wages (in rupees)* | *Workers* |
|---|---|
| 12.50—17.50 | 3 |
| 17.50—22.50 | 6 |
| 22.50—27.50 | 8 |
| 27.50—32.50 | 11 |
| 32.50—37.50 | 15 |
| 37.50—42.50 | 18 |
| 42.50—47.50 | 16 |
| 47.50—52.50 | 12 |
| 52.50—57.50 | 8 |
| 57.50—62.50 | 3 |
| | 100 |

6. Find the average marks scored by students from the following data :

| *Marks* | *Number of students* |
|---|---|
| Less than 10 | 3 |
| Less than 20 | 5 |
| Less than 30 | 9 |
| Less than 40 | 15 |
| Less than 50 | 20 |
| Less than 60 | 26 |
| Less than 70 | 34 |
| Less than 80 | 41 |
| Less than 90 | 45 |
| Less than 100 | 47 |

7. The following are the prices of certain important commodities required by a family. The importance (weight) attached to each commodity by the family is also indicated. Compare the mean price per kg with the weighted mean price per kg.

| *Commodity* | *Importance (Weight)* | *Price per kg* |
|---|---|---|
| Rice | 3 | Rs 8.50 |
| Wheat | 1 | Rs 3.20 |
| Oils | 2 | Rs 25.00 |
| Dals | 1 | Rs 7.25 |

## DEMERITS OF ARITHMETIC MEAN

Consider a small establishment where there are 7 employees : a supervisor, an accountant and 5 labourers. The labourers draw Rs 325, Rs 330, Rs 335, Rs 340, Rs 345 respectively; the supervisor draws Rs 1500 and the accountant Rs 1100 per month. The average wage of employees

$$= \text{Rs } \frac{325 + 330 + 335 + 340 + 345 + 1500 + 1100}{7}$$

$$= \text{Rs } \frac{4275}{7}$$

$$= \text{Rs } 610.70 \text{ (approx.)}$$

Clearly it does not give even an approximate estimate of any one of the employees.

This drawback of arithmetic mean and the consequent mis-representation of the results by the presence of a few extreme values in the data motivates us to look for another measure which is unaffected by the few extreme items.

*Remarks* :

(*a*) Arithmetic mean possesses all the characteristics which an ideal measure of location is expected to have.

(*b*) Arithmetic mean is very much affected by extreme values.

(*c*) Arithmetic mean cannot be obtained if a single observation is missing or lost or is illegible.

(*d*) Arithmetic mean cannot be calculated if we are dealing with qualitative characteristics which cannot be measured quantitatively such as intelligence, beauty, etc.

## 16.3 Median

**Median of a distribution is that value of the variable which divides it into two equal parts.** That is, it is that value of the variable which is such that number of observations before it is equal to the number of observations after it. In other words, **median is the value of the middle-most item.** Median is, thus, a **positional average.**

In the case of raw data or ungrouped data, after arranging the values of the variable in ascending or descending order of magnitude, the median is the middle value if the number of observations is odd, and if the number of observations is even, median is the arithmetic mean of the middle two terms.

For example, the median of the values 12, 15, 19, 22, 25 is 19 and median of 15, 13, 17, 14, 16, 10 i.e. of 10, 13, 14, 15, 16, 17 (after arranging in ascending order) is $\frac{14+15}{2} = 14.5$.

**Example 1 :** The following goals are scored by a team in a series of 6 matches :

1 2 5 3 4 0

Find the median.

**Solution :** The goals scored arranged in ascending order are :

0 1 2 3 4 5

There are 6 observations and there are two middle terms viz. $\frac{6}{2}$th and $\left(\frac{6}{2}+1\right)$th i.e. 3rd and 4th terms.

The median $= \frac{2+3}{2} = 2.5$

**Example 2 :** The weights (in kg) of 15 children are as follows :

31, 35, 27, 29, 32, 43, 37, 41, 34, 28, 36, 44, 45, 42, 30

Find the median.

**Solution :** Arranging the weights (in kg) in descending order, we get

45, 44, 43, 42, 41, 37, 36, 35, 34, 32, 31, 30, 29, 28, 27

There are 15 observations. Therefore, the middle-most item is $\frac{15+1}{2}$th item, i.e. 8th item. Its value is 35.

$\therefore$ Median is 35.

**Example 3 :** Find the median of the following frequency distribution:

| $x$ | 5 | 7 | 9 | 12 | 14 | 17 | 19 | 21 |
|---|---|---|---|---|---|---|---|---|
| $f$ | 6 | 5 | 3 | 6 | 5 | 3 | 2 | 4 |

**Solution :** We first compute the cumulative frequencies which will enable us to locate the middle item.

| $x$ | $f$ | *Cumulative frequency* ($c$) |
|---|---|---|
| 5 | 6 | 6 |
| 7 | 5 | 11 |
| 9 | 3 | 14 |
| 12 | 6 | 20 |
| 14 | 5 | 25 |
| 17 | 3 | 28 |
| 19 | 2 | 30 |
| 21 | 4 | 34 |
| | 34 | |

There are 34 observations. So the median will be the average of $\frac{34}{2} = 17$th and 18th observations. The value of 14th observation is 9. The value of each observation from 15th to 20th (which are 6 in number) is 12.

$\therefore$ The value of 17th observation is 12 and same is the value of the 18th observation.

$$\therefore \text{ Median is } \frac{12+12}{2} = 12$$

**Remarks :** Suppose there are 11 candidates : $A_i$, $i = 1, 2, ..., 11$ taking part in beauty contest. The contest is held in 6 phases. A candidate who qualifies in all the phase-tests is declared the beauty queen. The observations of the various phase-tests are as follows :

$A_2$ and $A_3$ are rejected in the I phase,
$A_5$ and $A_7$ are rejected in the II phase,
$A_{10}$ is rejected in the III phase,
$A_8$ is rejected in the IV phase,
$A_1$ and $A_{11}$ are rejected in the V phase,
$A_4$ and $A_9$ are rejected in the VI phase,
$A_6$ is declared the beauty queen.

Arranging the candidates in ascending order of their beauty levels, we get

$$A_2A_3,\ A_5A_7,\ A_{10},\ A_8,\ A_1A_{11},\ A_4A_9,\ A_6$$

Clearly beauty level of $A_8$ can be taken as the average level of beauty. Obviously this level is the median as number of candidates above and below this level is equal viz. 5. Notice that arithmetic mean cannot be obtained in this case. Thus, **median is the only measure to be used while dealing with qualitative data which cannot be measured quantitatively but still can be arranged in ascending or descending order.**

## 16.4 Mode

By the statement 'item $A$ is the most popular item' we mean that majority of the customers buy item $A$. The most popular item $A$ is referred to as mode. **Mode is that value of the variable which occurs most frequently.** In other words, mode is the value of the variable which is most predominent in the series. **Thus, in the case of a discrete frequency distribution, mode is the value of the variable corresponding to the maximum frequency.**

**Example :** Find the mode of the following frequency distribution:

| Size of shirt (in inches) | 28 | 30 | 32 | 34 | 36 | 38 |
|---|---|---|---|---|---|---|
| Number of buyers | 310 | 401 | 415 | 345 | 390 | 301 |

**Solution :** Maximum frequency is 415. Observation corresponding to this is 32. Therefore, mode is 32″. Thus, the shopkeeper must place the biggest order for shirts of size 32″.

*Remarks* :

(*a*) If the maximum frequency is repeated or when the maximum frequency occurs in the very beginning or at the end of the distribution or if there are irregularities in the distribution, the value of the mode is determined by other methods which are beyond the scope of the present book.

(*b*) Mode is the average used to find the ideal size in the manufacture of ready-made garments, shoes, etc.

(*c*) A frequency distribution $(x_i, f_i)$, $i = 1, 2, \ldots, n$ is said to be symmetrical when the variable values equidistant from the mean $(\bar{x})$ have equal frequencies i.e. when frequencies are symmetrically distributed about the mean. Thus for a symmetrical frequency distribution $(x_i, f_i)$, $i = 1, 2, \ldots, n$ the frequencies of the variable $(\bar{x} - y)$ and $(\bar{x} + y)$, which are at a distance $y$ from $\bar{x}$, are same.

For example, the following distribution is symmetrical about its mean $\bar{x} = 4$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $f$ : | 5 | 8 | 9 | 10 | 9 | 8 | 5 |

It may be noted that for a symmetrical distribution having a single mode

$$\text{Mean} = \text{Median} = \text{Mode}$$

A distribution which is not symmetrical is called asymmetrical distribution or skew distribution. For a moderately asymmetrical distribution,

$$\text{Mode} = 3\ \text{Median} - 2\ \text{Mean}$$

## Exercises 16.3

1. The points scored by a basket-ball team in a series of matches are as follows :
   15, 3, 8, 10, 22, 5, 27, 11, 12, 19, 18, 21, 13, 14
   Find the median.
2. The heights (in cm) of 15 students of Class X are :
   152, 147, 156, 149, 151, 159, 148, 160, 153, 154, 150, 143, 155, 157, 161
   Find the median.
3. Calculate the median of the following data relating to the selling price of an article in a sample of shops:

| Selling price (in rupees) | 5.00 | 5.05 | 5.10 | 5.15 | 5.20 | 5.25 |
|---|---|---|---|---|---|---|
| Number of shops | 11 | 12 | 43 | 35 | 40 | 20 |

4. Find the median for the following frequency distribution:

| $x$ | 6 | 7 | 5 | 2 | 10 | 9 | 3 |
|---|---|---|---|---|---|---|---|
| $f$ | 9 | 12 | 8 | 13 | 11 | 14 | 7 |

5. A drama was staged to raise funds for the school library. In a random sample of 100 tickets sold, the following observations were made :

| Cost of ticket (in rupees) | 1 | 3 | 5 | 10 | 25 | 50 |
|---|---|---|---|---|---|---|
| Number of tickets sold | 12 | 13 | 25 | 35 | 10 | 5 |

   Tickets of which denomination were printed in the largest quantity ?

6. Find the mode for the following frequency table :

| Weight (in kg) | 40 | 43 | 46 | 49 | 52 | 55 |
|---|---|---|---|---|---|---|
| Number of students | 5 | 8 | 16 | 9 | 7 | 3 |

7. In a test given to 15 candidates, the following marks (out of 100) are awarded. Which average will be a good representative of the data and why ? Calculate it. Find two other measures of location also.

52, 56, 49, 56, 48, 53, 54, 56, 52, 59, 53, 56, 54, 57, 56

8. Find mean, median and mode for the following frequency distribution;

| $x$ | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 |
|---|---|---|---|---|---|---|---|---|---|
| $f$ | 6 | 11 | 13 | 18 | 21 | 18 | 13 | 11 | 6 |

# ANSWERS

## Exercises 1.2

1. Maximise
   $P = 50x + 15y$
   subject to
   $250x + 50y \leqslant 5000$
   $x + y \leqslant 60$
   $x \geqslant 0, y \geqslant 0$
2. Minimise
   $P = 5x + 7y$
   subject to
   $2x + y \geqslant 8$
   $x + 2y \geqslant 10$
   $x \geqslant 0, y \geqslant 0$
3. Maximum value is 18 at (6,2)
4. Maximum value is 382 at (10, 38)
5. Minimum at (2, 3) and minimum value is 5.
6. Rs 1000 in Savings certificates and Rs 11000 in National Savings Bonds, Rs 1180
7. First class tickets : 40
   Economy class tickets : 160
   Maximum profit : Rs 64000
8. There are several solutions.
   One of the solutions is :
   Number of bracelets : 8
   Number of necklaces : 16
9. Cord type telephones : 200
   Cordless telephones : 100
10. Fertilizer type I : 100 kg
    Fertilizer type II : 80 kg
    Minimum cost : Rs 92

## Exercises 2.1

1. (*i*) Yes
   (*ii*) No
   (*iii*) Yes
   (*iv*) Yes
   (*v*) No
   (*vi*) Yes
2. (*i*) $x^6 + x^3 - 7x^2 + 4x$
   (*ii*) $-x^5 + 5x^3 - x^2 + 2x$
   (*iii*) $y^4 - 6y^3 + y^2 - 8y + 7$
   (*iv*) $y^6 - y^5 + y^4 - y^3 + y^2 - y$
   (*v*) $\frac{x}{4} - \frac{x^2}{3} - \frac{x}{2}$
3. (*i*), (*iv*), (*v*)
4. (*i*) binomial
   (*ii*) binomial
   (*iii*) trinomial
   (*iv*) monomial
   (*v*) monomial
   (*vi*) binomial
   (*vii*) monomial
   (*viii*) monomial
   (*ix*) trinomial

## Exercises 2.2

1. (*i*) $4x^2 - 5$ ; 2
   (*ii*) $x^2 + 4x - 2$ ; 2
   (*iii*) $2y^3 + 6y$ ; 3
   (*iv*) $12z^3 + 12z$ ; 3
   (*v*) $z^3 + 2z^2 + 4z - 1$ ; 3
   (*vi*) $-y^2 + 4y + 17$ ; 2
2. (*i*) $6y^2 + 2$ ; 2
   (*ii*) $4x^2$ ; 2
   (*iii*) $-x^4 - 4x^3 - 3x^2 - 4x - 1$ ; 4
   (*iv*) $y^4 - 4y^3 + 5y^2 - 6y$ ; 4
3. $x^3 + 3$
4. $y^3 + y^2 + 8y - 2$
5. (*i*) $x^2 + 10x + 24$ ; 2
   (*ii*) $y^2 + 5yz + 6z^2$ ; 2
   (*iii*) $x^3 - 3x^2 + 3x - 1$ ; 3
   (*iv*) $x^3 + 6x^2 + 11x + 6$ ; 3
   (*v*) $x^4 - x^3 + x^2 - 1$ ; 4
   (*vi*) $x^4 + 2x^3y + 2x^2y^2 + 2xy^3 + y^4$ : 4
   (*vii*) $x^4 + x^2 + 1$ ; 4
   (*viii*) $x^3 + y^3 + z^3 - 3xyz$ ; 3
6. (*i*) $x^4 - 1$
   (*ii*) $y^3 + 9y^2 + 23y + 15$

## Exercises 2.3

1. $x^2 - 3x + 2$ ; 0
2. $x + 1$, 0
3. $x^2 + 1$ ; 18
4. $x^2 + 3x + 1$ ; 13
5. $x^3 + x^2 + x + 1$ ; 2
6. $x^4 - 2x^2 + 5x + 4$ ; $-3x - 5$
7. $x$ ; $2x^2 - x + 1$
8. 0 ; $x^3 + x^2 - x - 1$
9. $x^5 - 3x^4 + 9x^3 - 27x^2 + 84x - 252$ ; 766
10. $\frac{3x}{4} + \frac{1}{4}$ ; 1

## Exercises 2.4

1. (*i*) $-2$ (*ii*) 0
   (*iii*) $-6$
2. (*i*) 0 (*ii*) 6
   (*iii*) $-60$
3. (*i*) 0 (*ii*) $-12$
   (*iii*) 0

## Exercises 2.5

1. Yes
2. Yes
3. No
4. No
5. No
6. 1, $-1$
7. $p = 7$, $q = -18$

## Exercises 2.6

5. $-4$
7. (*i*) $-4$ (*ii*) 2 (*iii*) 1, $-1$, $-3$

## Exercises 3.1

1. (i) $(x-3)(x-4)$
(ii) $(2z-3)(3z-1)$
(iii) Polynomial cannot be factorised
(iv) Polynomial cannot be factorised
(v) $(x-1)(x-\sqrt{2})$
(vi) $(4x-7y)(2x+y)$
(vii) $3x(x-4)(2x+3)$
(viii) $(rx-p)(px+2q)$
(ix) $(px-3q)(x+4p)$
(x) $-2\left[x+\frac{3}{4}+\frac{\sqrt{65}}{4}\right]\left[x+\frac{3}{4}-\frac{\sqrt{65}}{4}\right]$
(xi) $[x+2+\sqrt{5}]\,[\sqrt{5}-2-x]$
(xii) $[x-5-3\sqrt{3}]\,[x+5+3\sqrt{3}]$
(xiii) $(aby+c)(aby+c)$
(xiv) $(a^2x+1)(b^2x-1)$
(xv) $(z^2+1)(z^2+8)$
(xvi) $(x^3+4y)(x^3+y)$
(xvii) $(x+\sqrt{3}y)(x-\sqrt{3}y)(x^2+5y^2)$
(xviii) $(x-1)(x+1)(x^2+6)$
(xix) $(x^n-2)(x^n+1)$
(xx) $(y^n+2)(y^n-1)$
2. Yes, it is possible
3. Yes, it is possible
4. $x+4$, $x$ and $x-6$
5. $2x$, $x+7$, $x-2$

## Exercises 3.2

1. (i) (0, 0); $y$-axis is the axis of symmetry
(ii) $(-1, -2)$; $x=-1$ is the axis of symmetry
(iii) (1, 2); $x=1$ is the axis of symmetry
(iv) (2, 1) ; $x=2$ is the axis of symmetry.
2. (i) Point of minimum $(-1, 4)$ ; axis of symmetry : $x=-1$
(ii) Point of maximum $(-2, 8)$ ; axis of symmetry : $x=-2$
(iii) Point of maximum (0, 0) ; axis of symmetry : $y$-axis
(iv) Point of minimum $(3, -7)$ ; axis of symmetry : $x=3$
3. 3.5 seconds
4. $x=\frac{3}{2}$
5. Maximum for 50 units
Maximum revenue : Rs 2500/-
6. $x=5$ doses
8. $x=6$ amps
Maximum power : 720 watts
9. (b) Rs 100

## Exercises 3.3

1. (*i*) 3, 5
   (*ii*) $1, -\frac{5}{3}$
   (*iii*) No solution
   (*iv*) $\frac{1}{2}, -\frac{1}{3}$
   (*v*) No solution
2. (*i*) $\frac{1}{2}, -\frac{1}{2}$
   (*ii*) $\frac{3}{4}$
   (*iii*) $-\frac{1}{5}, -\frac{4}{5}$
   (*iv*) $-a, b$
   (*v*) $0, -\frac{(a+b)}{3}$
3. (*i*) $\phi$
   (*ii*) $\left\{-3\sqrt{3}, \frac{-2\sqrt{3}}{3}\right\}$
   (*iii*) $\left\{\frac{\sqrt{3}}{4}, \frac{-2\sqrt{3}}{3}\right\}$
   (*iv*) $\left\{\frac{3q}{p}, -4p\right\}$
   (*v*) $\left\{\frac{1}{m+n}, \frac{-2}{m+n}\right\}$
4. (*i*) $4, -4$
   (*ii*) $2\sqrt{2}, -2\sqrt{2}$
   (*iii*) $-\frac{1}{8}$
   (*iv*) 0
   (*v*) $-\frac{1}{2}$
5. (*i*) $k < 4$ (*ii*) $k = 4$
6. (*i*) $\lambda > 5$ or $\lambda < -1$
   (*ii*) 5 or $-1$

## Exercises 3.4

1. $\pm 3, \pm 2$
2. $\pm\sqrt{\frac{5}{3}}$
3. $1, \frac{3 \pm \sqrt{17}}{2}$
4. 1, 2, 3
5. $-1, 1, 1 \pm \sqrt{2}$
6. $-1, 1, \frac{3 \pm \sqrt{\ }}{2}$
7. $-\frac{1}{2}, 2, -1 \pm \sqrt{2}$
8. $\frac{1}{2}, 2$
9. 0
10. $-4$
11. $3, -3$
12. $\frac{1}{2}, \frac{4}{3}$
13. 1
14. $\frac{4}{13}, \frac{9}{13}$
15. $\frac{3}{5}$
16. 4
17. 10
18. $\phi$
19. $\phi$
20. $\frac{3}{2}, -2$

## Exercises 3.5

1. 9, 11 and −9, −11
2. $20 + 10\sqrt{3}, 20 - 10\sqrt{3}$
3. 8, 9 and 10
4. 24
5. 21 seats
6. 6 points
7. 15 m and 20 m
8. 10 cm, 8 cm and 6 cm
9. 20 km/hr, 15 km/hr
10. 5 m
11. 4.3 km/hr
12. 3 km/hr
13. Ashu : 5 years and Mrs. Veena : 25 years
14. Father : 36 years, Son : 9 years
15. 9 and 6. Also 9 and − 6
16. 3 cm and 9 cm
17. Side of the larger square : 8 cm
    Side of the smaller square : 5 cm

## Exercises 4.1

1. $(x + 1)(x^2 - x + 1)$
2. $(6 + a)(36 - 6a + a^2)$
3. $a(a + 1)(a^2 - a + 1)$
4. $(2x + 5y)(4x^2 - 10xy + 25y^2)$
5. $(3a + \sqrt{7}\,b)(9a^2 - 3\sqrt{7}\,ab + 7b^2)$
6. $\left(\frac{a}{2} + \sqrt{5}\,b\right)\left(\frac{a^2}{4} - \frac{ab\sqrt{5}}{2} + 5b^2\right)$
7. $\left(a + \frac{y}{7}\right)\left(a^2 - \frac{ay}{7} + \frac{y^2}{49}\right)$
8. $\left(\frac{a}{\sqrt{2}} + \frac{b}{\sqrt{3}}\right)\left(\frac{a^2}{2} - \frac{ab}{\sqrt{6}} + \frac{b^2}{3}\right)$
9. $(5x - y)(7x^2 - 10xy + 19y^2)$
10. $(y^2 + z^2)(y^4 - y^2z^2 + z^4)$
11. $(5y^{\frac{1}{4}} + z^{\frac{1}{4}})(25y^{\frac{1}{2}} - 5y^{\frac{1}{4}}z^{\frac{1}{4}} + z^{\frac{1}{2}})$
12. $(\sqrt{5}\,x^{\frac{1}{2}} + 2y^{\frac{1}{2}})(5x - 2\sqrt{5}\,x^{\frac{1}{2}}y^{\frac{1}{2}} + 4y)$
13. $9(a + b)(a^2 + ab + b^2)$
14. $x(y + 9x)(y^2 - 9xy + 81x^2)$

## Exercises 4.2

$(a - 2b)(a^2 + 2ab + 4b^2)$

2. $2(a - 2)(a^2 + 2a + 4)$
3. $(3 - 2x)(9 + 6x + 4x^2)$
4. $2(5x - 2y)(25x^2 + 10xy + 4y^2)$

5. $(\sqrt{5}\,a - 7)\,(5\,a^2 + 7\sqrt{5}\,a + 49)$
6. $(\sqrt{3}\,x - \sqrt{5}\,y)\,(3x^2 + \sqrt{15}\,xy + 5y^2)$
7. $(a-2)\,(a+2)\,(a^2 - 2\,a + 4)\,(a^2 + 2\,a + 4)$
8. $(x-y)\,(x+y)\,(x^2 - xy + y^2)\,(x^2 + xy + y^2)$
9. $(2\,x^2 - y)\,(4\,x^4 + 2\,x^2\,y + y^2)$
10. $(x^{\frac{1}{2}} - y^{\frac{1}{2}})\,(x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y)$
11. $ab\,(b-2a)\,(b^2 + 2ba + 4a^2)$

## Exercises 4.3

1. (i) $(a - b + c)\,(a^2 + b^2 + c^2 + ab + bc - ca)$
   (ii) $(a - 2\,b + 5\,c)\,(a^2 + 4\,b^2 + 25\,c^2 + 2ab + 10bc - 5ac)$
   (iii) $\left(2\,a - \frac{b}{2} - c\right)\left(4\,a^2 + \frac{b^2}{4} + c^2 + ab + 2\,ac - \frac{bc}{2}\right)$
   (iv) $(\sqrt{2}\,a + b + c)\,(2\,a^2 + b^2 + c^2 - \sqrt{2}\,ab - \sqrt{2}\,ac - bc)$
2. (i) $(b + c)\,(7\,b^2 - bc + c^2)$
   (ii) $(b + c + 1)\,(b^2 + c^2 + 1 - bc - b - c)$
   (iii) $(a + 5\,c - 2)\,(a^2 + 25c^2 + 4 + 2a + 10c - 5ac)$
   (iv) $(\sqrt{2} + 1 + c)\,(3 + c^2 - \sqrt{2} - c - \sqrt{2}c)$
4. (i) $3xyz$ (ii) $6xy$ (iii) $6ab$

## Exercises 4.4

1. (i) $(a-1)\,(a+1)\,(a^2+1)$
   (ii) $(2a - 5b)\,(2a + 5b)\,(4a^2 + 25b^2)$
   (iii) $a^2b^2\,(b-a)\,(b+a)\,(b^2+a^2)$
   (iv) $(x-y)\,(x+y)\,(x^2+y^2)\,(x^4+y^4)$
   (v) $(x^{\frac{1}{5}} - y^{\frac{1}{5}})\,(x^{\frac{1}{5}} + y^{\frac{1}{5}})\,(x^{\frac{2}{5}} + y^{\frac{2}{5}})$
   (vi) $(x^{\frac{1}{3}} - 3y^{\frac{1}{3}})\,(x^{\frac{1}{3}} + 3y^{\frac{1}{3}})\,(x^{\frac{2}{3}} + 9y^{\frac{2}{3}})$
   (vii) $(x - \sqrt{3}y)\,(x + \sqrt{3}y)\,(x^2 + 3y^2)$

## Exercises 4.5

1. (i) $x - 3$
   (ii) $2\,(x+2)$
   (iii) $x - 4$
   (iv) $x^2 - 6x + 9$
   (v) $4\,(x-1)^2$
   (vi) $x^3 + x^2 + x + 1$
2. (i) $x - 3$
   (ii) $5\,x - 1$
   (iii) $x^2 - 4x + 3$
   (iv) $2(x^2+1)$
   (v) $x^2 + 4$

3. (i) $2(x^4-11x^3+41x^2-61x+30)$
(ii) $x^3-x^2-x+1$
(iii) $5x^3-15x+10$
(iv) $x^3-12x^2+44x-48$
(v) $4x^3-36x^2+92x-60$
(vi) $2x^4-2$
(vii) $x^5+2x^4y-xy^4-2y^5$
(viii) $x^4-6x^3y+13x^2y^2-12xy^3+4y^4$
(ix) $2x^4-18x^3+42x^2+2x-60$
(x) $x^4-2x^3-24x^2-5x+30$

## Exercises 5.1

1. Rs 337.50
2. Rs 750
3. Rs 25
4. Rs 2500
5. Rs 2970
6. Rs 2100
7. Rs 320
8. Rs 120
9. Rs 162
10. Rs 672
11. Rs 7.50
12. Rs 100
13. Annual income : Rs 320 ; $5\frac{1}{3}\%$
14. 1500 shares, Rs 625
15. Rs 20800

## Exercises 5.2

1. Second is better
2. Second is better
3. Second is better. Investment = Rs 3660
4. Rs 180 increase
5. Rs 12500, Rs $\frac{80000}{3}$ respectively
6. Total investment = Rs 8460
7. Rs 3000, Rs 6000
8. Rs 26
9. Rs 100
10. Rs 105 (Increase)
11. Rs 469.60
12. Rs 740
13. Rs 12000
14. Rs 75000
15. Rs 72

## Exercises 6.1

1. 40% per annum
2. $68\frac{4}{7}\%$ per annum
3. 75% per annum
4. 47% per annum
5. $33\frac{1}{3}\%$ per annum
6. 47% per annum
7. 30% per annum
8. (i) 54.5% per annum
   (ii) 24% per annum
   (iii) 41.7% per annum
   (iv) 21.8% per annum
9. Rs 20
10. Rs 26
11. Rs 130

## Exercises 6.2

1. 1st Instalment : Rs 1848
   2nd Instalment : Rs 1736
   3rd Instalment : Rs 1624
   4th Instalment : Rs 1512
2. 1st Instalment : Rs 2600
   2nd Instalment : Rs 2400
   3rd Instalment : Rs 2200
3. 1st Instalment : Rs 3598
   2nd Instalment : Rs 3332
   3rd Instalment : Rs 3066
4. Rs 463.05
5. Rs 1352
6. Rs 1458
7. Rs 486
8. Rs 1655
9. 1st Instalment = Rs 1450, interest = Rs 232
   2nd Instalment = Rs 1250, interest = Rs 432
10. Principal = Rs 4550, interest = Rs 1930
11. Sum borrowed = Rs 12610, interest = Rs 1281.50
12. Rs 6000, interest = Rs 1200
13. Rs 2048
14. Rs 1900
15. Rs 1450
16. Cash price = Rs 6000
    Total interest = Rs 790
17. Cash price = Rs 800
    Total interest = Rs 260

## Exercises 7.1

1. Rs 140
2. April : Rs 1550
   May : Rs 1760
3. June : Rs 500
   July : Rs 1500
4. Interest : Rs 70.83
   Balance : Rs 2870.83
5. Rs 10132.08
6. 5.42, 1455.42
7. 51.25, 1451.25
8.

| *Date* | *Particulars* | *Withdrawals* *Rs P* | *Deposits* *Rs P* | *Balance* *Rs P* |
|---|---|---|---|---|
| *1983* | | | | |
| August 1 | By cash | | 100.00 | 100.00 |
| Sept. 2 | By cash | | 100.00 | 200.00 |
| Oct. 1 | By cash | | 100.00 | 300.00 |
| Oct. 3 | By interest | | 1.25 | 301.25 |
| Oct. 17 | To cheque | 200.00 | | 101.25 |
| Nov. 2 | By cash | | 100.00 | 201.25 |
| Dec. 1 | By cash | | 100.00 | 301.25 |
| *1984* | | | | |
| Jan. 2 | By cash | | 100.00 | 401.25 |
| Jan. 18 | To cheque | 200.00 | | 201.25 |
| Feb. 1 | By cash | | 100.00 | 301.25 |
| Oct. 1 | By interest | | 13.40 | 314.65 |

9. Rs 3043.75
10. Rs 39.38 ; Rs 2739.38

## Exercises 7.2

1. Rs 1472.80
2. Rs 5054.40, Rs 4001.50
3. Rs 6318.00
4. 60
5. 120
6. 4 years
7. Rs 50
8. Rs 80

## Exercises 8.1

1. *Nil*
2. *Rs 1400*
3. Rs 1810
4. Rs 5200
5. *Rs 900*
6. *Rs 42650*
7. *Rs 636*
8. *Rs 560*

## Exercises 8.2

1. Rs 395.90
2. Rs 56.75
3. Rs 1142.20
4. Rs 8900
5. Rs 90

## Exercises 9.1

1. (*i*) Yes
   (*ii*) Yes
   (*iii*) No
   (*iv*) No, Corresponding angles are not equal
2. AC = 18 cm, DE = 4 cm
4. 8, 11
5. (*i*) Yes (*ii*) Yes (*iii*) No (*iv*) Yes (*v*) No
6. $x = 6$
10. 90 cm
11. 6 cm
17. 1 : 4

## Exercises 9.2

1. (*iii*)
3. 12 m
4. 13 m
5. $10\sqrt{10}$ m

## Exercises 10.1

1. (*i*) 3 cm
   (*ii*) 5 cm
   (*iii*) 10 cm or 3 cm
2. (*i*) 5 cm
   (*ii*) 2 cm
   (*iii*) 11 cm
5. (*a*) 42°, 138° (*b*) 75°
15. (*i*) 30°
   (*ii*) 90°
   (*iii*) 60°
   (*iv*) 90°
   (*v*) 30°

## Exercises 11.1

1. Centre of the circle passing through the three given points.
2. $\phi$
3. Sphere with centre O
4. (i) AB = 7 cm and AC = 3 cm, AB = 6 cm and AC = 4 cm, AB = 5 cm and AC = 5 cm
   (ii) No. In this case AC + BC < AB
   (iii) AB = 8 cm and AC = 2 cm, AB = 1 cm and AC = 9 cm, etc.

## Exercises 11.2

1. Line parallel to the given lines such that its distance from both the lines is the same.
2. Perpendicular bisector of BC, yes
3. (i) If any side of a quadrilateral formed by the four points subtends equal angles at the remaining vertices.
   (ii) If any side of the $n$-sided polygon subtends equal angles at the remaining vertices.

## Exercises 11.3

5. Circle with centre of the given circle as centre and $R + r$ as radius.
6. Circle with centre of the given circle as centre and $R - r$ as radius.

## Exercises 13.1

1. $\tan\theta = \frac{12}{5}$, $\cos\theta = \frac{5}{13}$
2. $\cos\theta = \frac{21}{29}$, $\sin\theta = \frac{20}{29}$
3. $-\frac{1}{5}$
4. $-\frac{169}{96}$
5. $\cot\theta = \frac{p}{\sqrt{q^2-p^2}}$, $\sin\theta = \frac{\sqrt{q^2-p^2}}{q}$
15. $\frac{b+a}{b-a}$

## Exercises 13.2

1. (i) 4
   (ii) $\frac{7}{4}$
   (iii) $-\frac{1}{3}$
   (iv) 5
   (v) 0
   (vi) 1
   (vii) 1
2. (i) 1
   (ii) $\sqrt{3}-1$
   (iii) $\frac{3(\sqrt{3}-1)}{4}$
3. 9

## Exercises 13.3

3. (i) $\frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$
   (ii) $\frac{\sqrt{6}-\sqrt{2}}{4}$

5. (i) $\frac{13}{6}$
   (ii) $\frac{17}{4}$

6. (i) 25
   (ii) 12.48

## Exercises 14.1

1. (i) .8746
   (ii) .6018
   (iii) 1.335
   (iv) .7679
   (v) .1334
   (vi) .9997
   (vii) .7116
   (viii) 2.424

2. (i) 14°30′
   (ii) 23°40′
   (iii) 47° 10′
   (iv) 57° 50′
   (v) 63° 30′
   (vi) 69° 20′
   (vii) 44° 26′ (approx.)
   (viii) 67° 47′ (approx.)

## Exercises 14.2

1. 86.60 m
2. 285.86 m
3. 30°
4. 71.62 m
5. 2.85 m
6. 401.95 m
7. 1732 m
8. 25 m from one side of the road, $h = 25\sqrt{3}$ m
9. 3.66 m
10. 7.04 m
11. 126.8 m
12. 204.8 m

## Exercises 15.1

1. 3744 $cm^2$
2. 3234 $cm^3$, 3516 $cm^3$
3. Rs 27.97
4. $\frac{1}{2}$m
5. Rs 640
6. $\frac{1}{2}$ m
7. 2250000 $m^2$
8. 8 m, 12 m, 16 m

## Exercises 15.2

1. Rs 110
2. $269\frac{1}{2}$ $cm^3$
3. 2.096 kg
4. $404\frac{1}{4}$ cubic units
5. 6.8 m (approx.)
6. 44 m ; 4620 $m^3$
7. 4 m
8. 1408 $cm^2$ ; 3850 $cm^3$
9. Rs 7128 ; Rs 3300
10. 787.5 cm

## Exercises 11.1

1. Centre of the circle passing through the three given points.
2. $\phi$
3. Sphere with centre O
4. (*i*) AB = 7 cm and AC = 3 cm, AB = 6 cm and AC = 4 cm, AB = 5 cm and AC = 5 cm
   (*ii*) No. In this case AC + BC < AB
   (*iii*) AB = 8 cm and AC = 2 cm, AB = 1 cm and AC = 9 cm, etc.

## Exercises 11.2

1. Line parallel to the given lines such that its distance from both the lines is the same.
2. Perpendicular bisector of BC, yes
3. (*i*) If any side of a quadrilateral formed by the four points subtends equal angles at the remaining vertices.
   (*ii*) If any side of the *n*-sided polygon subtends equal angles at the remaining vertices.

## Exercises 11.3

5. Circle with centre of the given circle as centre and $R + r$ as radius.
6. Circle with centre of the given circle as centre and $R - r$ as radius.

## Exercises 13.1

1. $\tan \theta = \frac{12}{5}$, $\cos \theta = \frac{5}{13}$
2. $\cos \theta = \frac{21}{29}$, $\sin \theta = \frac{20}{29}$
3. $-\frac{1}{5}$
4. $-\frac{169}{96}$
5. $\cot \theta = \frac{p}{\sqrt{q^2 - p^2}}$, $\sin \theta = \frac{\sqrt{q^2 - p^2}}{q}$
15. $\frac{b+a}{b-a}$

## Exercises 13.2

1. (*i*) 4
   (*ii*) $\frac{7}{4}$
   (*iii*) $-\frac{1}{3}$
   (*iv*) 5
   (*v*) 0
   (*vi*) 1
   (*vii*) 1
2. (*i*) 1
   (*ii*) $\sqrt{3} - 1$
   (*iii*) $\frac{3(\sqrt{3}-1)}{4}$
3. 9

## Exercises 13.3

3. (*i*) $\frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$

   (*ii*) $\frac{\sqrt{6}-\sqrt{2}}{4}$

5. (*i*) $\frac{13}{6}$

   (*ii*) $\frac{17}{4}$

6. (*i*) 25

   (*ii*) 12.48

## Exercises 14.1

1. (*i*) .8746
   (*ii*) .6018
   (*iii*) 1.335
   (*iv*) .7679
   (*v*) .1334
   (*vi*) .9997
   (*vii*) .7116
   (*viii*) 2.424

2. (*i*) 14°30′
   (*ii*) 23°40′
   (*iii*) 47° 10′
   (*iv*) 57° 50′
   (*v*) 63° 30′
   (*vi*) 69° 20′
   (*vii*) 44° 26′ (approx.)
   (*viii*) 67° 47′ (approx.)

## Exercises 14.2

1. 86.60 m
2. 285.86 m
3. 30°
4. 71.62 m
5. 2.85 m
6. 401.95 m
7. 1732 m
8. 25 m from one side of the road, $h = 25\sqrt{3}$ m
9. 3.66 m
10. 7.04 m
11. 126.8 m
12. 204.8 m

## Exercises 15.1

1. 3744 $cm^2$
2. 3234 $cm^3$, 3516 $cm^3$
3. Rs 27.97
4. $\frac{1}{2}$ m
5. Rs 640
6. $\frac{1}{2}$ m
7. 2250000 $m^2$
8. 8 m, 12 m, 16 m

## Exercises 15.2

1. Rs 110
2. $269\frac{1}{2}$ $cm^3$
3. 2.096 kg
4. $404\frac{1}{4}$ cubic units
5. 6.8 m (approx.)
6. 44 m ; 4620 $m^3$
7. 4 m
8. 1408 $cm^2$ ; 3850 $cm^3$
9. Rs 7128 ; Rs 3300
10. 787.5 cm

## Exercises 15.3

1. 25 cm
2. 198 $cm^2$
3. Rs 2464; Rs 4.40
4. 15 cm
5. 13 cm, 5 cm
6. 594 kg
7. $251\frac{3}{7}$ m
8. 61600 $m^3$
9. 20 cm
10. Rs 29700

## Exercises 15.4

1. 84000
2. 120.9 cm
3. $\frac{5}{6}$ cm
4. 7 cm
5. 103.62 $cm^2$
6. $6(19)^{\frac{1}{3}}$ cm
7. 512
8. 36 cm
9. $\frac{1}{2}\left(181\right)^{\frac{1}{3}}$ cm

## Exercises 16.1

1. 32 marks
2. 6.33 (approx.)
3. Rs 58.31 (approx.)
4. 28.58 marks (approx.)
5. 154 cm
6. Rs 1445
7. 52.75 kg
8. Rs 1500.00
9. 28
10. 236

## Exercises 16.2

1. 55.8
2. Rs 1426.25 (approx.)
3. 64.4
4. Rs 1069.68 (approx.)
5. Rs 38.80
6. 52.87 (approx.)
7. Mean = Rs 10.99 per kg
   Weighted mean = Rs 12.28 per kg

## Exercises 16.3

1. 13.5
2. 153 cm
3. Rs 5.15
4. 6.5
5. Tickets of Rs 10
6. 46 kg
7. Mean ; Mean = 54.07 (approx.)
   Median = 54
   Mode = 56
8. Mean = Median = Mode = 11

# Appendix : VALUES OF TRIGONOMETRIC RATIOS

| Degrees | sin | cos | tan | Degrees | sin | cos | tan |
|---|---|---|---|---|---|---|---|
| **0°00′** | .0000 | 1.0000 | .0000 | **6°00′** | .1045 | .9945 | .1051 |
| 10 | .0029 | 1.0000 | .0029 | 10 | .1074 | .9942 | .1080 |
| 20 | .0058 | 1.0000 | .0058 | 20 | .1103 | .9939 | .1110 |
| 30 | .0087 | 1.0000 | .0087 | 30 | .1132 | .9936 | .1139 |
| 40 | .0116 | .9999 | .0116 | 40 | .1161 | .9932 | .1169 |
| 50 | .0145 | .9999 | .0145 | 50 | .1190 | .9929 | .1198 |
| **1°00′** | .0175 | .9998 | .0175 | **7°00′** | .1219 | .9925 | .1228 |
| 10 | .0204 | .9998 | .0204 | 10 | .1248 | .9922 | .1257 |
| 20 | .0233 | .9997 | .0233 | 20 | .1276 | .9918 | .1287 |
| 30 | .0262 | .9997 | .0262 | 30 | .1305 | .9914 | .1317 |
| 40 | .0291 | .9996 | .0291 | 40 | .1334 | .9911 | .1346 |
| 50 | .0320 | .9995 | .0320 | 50 | .1363 | .9907 | .1376 |
| **2°00′** | .0349 | .9994 | .0349 | **8°00′** | .1392 | .9903 | .1405 |
| 10 | .0378 | .9993 | .0378 | 10 | .1421 | .9899 | .1435 |
| 20 | .0407 | .9992 | .0407 | 20 | .1449 | .9894 | .1465 |
| 30 | .0436 | .9990 | .0437 | 30 | .1478 | .9890 | .1495 |
| 40 | .0465 | .9989 | .0466 | 40 | .1507 | .9886 | .1524 |
| 50 | .0494 | .9988 | .0495 | 50 | .1536 | .9881 | .1554 |
| **3°00′** | .0523 | .9986 | .0524 | **9°00′** | .1564 | .9877 | .1584 |
| 10 | .0552 | .9985 | .0553 | 10 | .1593 | .9872 | .1614 |
| 20 | .0581 | .9983 | .0582 | 20 | .1622 | .9868 | .1644 |
| 30 | .0610 | .9981 | .0612 | 30 | .1650 | .9863 | .1673 |
| 40 | .0640 | .9980 | .0641 | 40 | .1679 | .9858 | .1703 |
| 50 | .0669 | .9978 | .0670 | 50 | .1708 | .9853 | .1733 |
| **4°00′** | .0698 | .9976 | .0699 | **10°00′** | .1736 | .9848 | .1763 |
| 10 | .0727 | .9974 | .0729 | 10 | .1765 | .9843 | .1793 |
| 20 | .0756 | .9971 | .0758 | 20 | .1794 | .9838 | .1823 |
| 30 | .0785 | .9969 | .0787 | 30 | .1822 | .9833 | .1853 |
| 40 | .0814 | .9967 | .0816 | 40 | .1851 | .9827 | .1883 |
| 50 | .0843 | .9964 | .0846 | 50 | .1880 | .9822 | .1914 |
| **5°00′** | .0872 | .9962 | .0875 | **11°00′** | .1908 | .9816 | .1944 |
| 10 | .0901 | .9959 | .0904 | 10 | .1937 | .9811 | .1974 |
| 20 | .0929 | .9957 | .0934 | 20 | .1965 | .9805 | .2004 |
| 30 | .0958 | .9954 | .0963 | 30 | .1994 | .9799 | .2035 |
| 40 | .0987 | .9951 | .0992 | 40 | .2022 | .9793 | .2065 |
| 50 | .1016 | .9948 | .1022 | 50 | .2051 | .9787 | .2095 |

## Appendix : VALUES OF TRIGONOMETRIC RATIOS

| Degrees | sin | cos | tan | Degrees | sin | cos | tan |
|---|---|---|---|---|---|---|---|
| **12°00** | .2079 | .9781 | .2126 | **18°00** | .3090 | .9511 | .3249 |
| 10 | .2108 | .9775 | .2156 | 10 | .3118 | .9502 | .3281 |
| 20 | .2136 | .9769 | .2186 | 20 | .3145 | .9492 | .3314 |
| 30 | .2164 | .9763 | .2217 | 30 | .3173 | .9483 | .3346 |
| 40 | .2193 | .9757 | .2247 | 40 | .3201 | .9474 | .3378 |
| 50 | .2221 | .9750 | .2278 | 50 | .3228 | .9465 | .3411 |
| **13°00′** | .2250 | .9744 | .2309 | **19°00′** | .3256 | .9455 | .3443 |
| 10 | .2278 | .9737 | .2339 | 10 | .3283 | .9446 | .3476 |
| 20 | .2306 | .9730 | .2370 | 20 | .3311 | .9436 | .3508 |
| 30 | .2334 | .9724 | .2401 | 30 | .3338 | .9426 | .3541 |
| 40 | .2363 | .9717 | .2432 | 40 | .3365 | .9417 | .3574 |
| 50 | .2391 | .9710 | .2462 | 50 | .3393 | .9407 | .3607 |
| **14°00′** | .2419 | .9703 | .2493 | **20°00′** | .3420 | .9397 | .3640 |
| 10 | .2447 | .9696 | .2524 | 10 | .3448 | .9387 | .3673 |
| 20 | .2476 | .9689 | .2555 | 20 | .3475 | .9377 | .3706 |
| 30 | .2504 | .9681 | .2586 | 30 | .3502 | .9367 | .3739 |
| 40 | .2532 | .9674 | .2617 | 40 | .3529 | .9356 | .3772 |
| 50 | .2560 | .9667 | .2648 | 50 | .3557 | .9346 | .3805 |
| **15°00′** | .2588 | .9659 | .2679 | **21°00′** | .3584 | .9336 | .3839 |
| 10 | .2616 | .9652 | .2711 | 10 | .3611 | .9325 | .3872 |
| 20 | .2644 | .9644 | .2742 | 20 | .3638 | .9315 | .3906 |
| 30 | .2672 | .9636 | .2773 | 30 | .3665 | .9304 | .3939 |
| 40 | .2700 | .9628 | .2805 | 40 | .3692 | .9293 | .3973 |
| 50 | .2728 | .9621 | .2836 | 50 | .3719 | .9283 | .4006 |
| **16°00′** | .2756 | .9613 | .2867 | **22°00′** | .3746 | .9272 | .4040 |
| 10 | .2784 | .9605 | .2899 | 10 | .3773 | .9261 | .4074 |
| 20 | .2812 | .9596 | .2931 | 20 | .3800 | .9250 | .4108 |
| 30 | .2840 | .9588 | .2962 | 30 | .3827 | .9239 | .4142 |
| 40 | .2868 | .9580 | .2994 | 40 | .3854 | .9228 | .4176 |
| 50 | .2896 | .9572 | .3026 | 50 | .3881 | .9216 | .4210 |
| **17°00′** | .2924 | .9563 | .3057 | **23°00′** | .3907 | .9205 | .4245 |
| 10 | .2952 | .9555 | .3089 | 10 | .3934 | .9194 | .4279 |
| 20 | .2979 | .9546 | .3121 | 20 | .3961 | .9182 | .4314 |
| 30 | .3007 | .9537 | .3153 | 30 | .3987 | .9171 | .4348 |
| 40 | .3035 | .9528 | .3185 | 40 | .4014 | .9159 | .4383 |
| 50 | .3062 | .9520 | .3217 | 50 | .4041 | .9147 | .4417 |

## Appendix : VALUES OF TRIGONOMETRIC RATIOS

| Degrees | sin | cos | tan | Degrees | sin | cos | tan |
|---|---|---|---|---|---|---|---|
| **24°00′** | .4067 | .9135 | .4452 | **30°00′** | .5000 | .8660 | .5774 |
| 10 | .4094 | .9124 | .4487 | 10 | .5025 | .8646 | .5812 |
| 20 | .4120 | .9112 | .4522 | 20 | .5050 | .8631 | .5851 |
| 30 | .4147 | .9100 | .4557 | 30 | .5075 | .8616 | .5890 |
| 40 | .4173 | .9088 | .4592 | 40 | .5100 | .8601 | .5930 |
| 50 | .4200 | .9075 | .4628 | 50 | .5125 | .8587 | .5969 |
| **25°00′** | .4226 | .9063 | .4663 | **31°00′** | .5150 | .8572 | .6009 |
| 10 | .4253 | .9051 | .4699 | 10 | .5175 | .8557 | .6048 |
| 20 | .4279 | .9038 | .4734 | 20 | .5200 | .8542 | .6088 |
| 30 | .4305 | .9026 | .4770 | 30 | .5225 | .8526 | .6128 |
| 40 | .4331 | .9013 | .4806 | 40 | .5250 | .8511 | .6168 |
| 50 | .4358 | .9001 | .4841 | 50 | .5275 | .8496 | .6208 |
| **26°00** | .4384 | .8988 | .4877 | **32°00′** | .5299 | .8480 | .6249 |
| 10 | .4410 | .8975 | .4913 | 10 | .5324 | .8465 | .6289 |
| 20 | .4436 | .8962 | .4950 | 20 | .5348 | .8450 | .6330 |
| 30 | .4462 | .8949 | .4986 | 30 | .5373 | .8434 | .6371 |
| 40 | .4488 | .8936 | .5022 | 40 | .5398 | .8418 | .6412 |
| 50 | .4514 | .8923 | .5059 | 50 | .5422 | .8403 | .6453 |
| **27°00′** | .4540 | .8910 | .5095 | **33°00′** | .5446 | .8387 | .6494 |
| 10 | .4566 | .8897 | .5132 | 10 | .5471 | .8371 | .6536 |
| 20 | .4592 | .8884 | .5169 | 20 | .5495 | .8355 | .6577 |
| 30 | .4617 | .8870 | .5206 | 30 | .5519 | .8339 | .6619 |
| 40 | .4643 | .8857 | .5243 | 40 | .5544 | .8323 | .6661 |
| 50 | .4669 | .8843 | .5280 | 50 | .5568 | .8307 | .6703 |
| **28°00′** | .4695 | .8829 | .5317 | **34°00′** | .5592 | .8290 | .6745 |
| 10 | .4720 | .8816 | .5354 | 10 | .5616 | .8274 | .6787 |
| 20 | .4746 | .8802 | .5392 | 20 | .5640 | .8258 | .6830 |
| 30 | .4772 | .8788 | .5430 | 30 | .5664 | .8241 | .6873 |
| 40 | .4797 | .8774 | .5467 | 40 | .5688 | .8225 | .6916 |
| 50 | .4823 | .8760 | .5505 | 50 | .5712 | .8208 | .6959 |
| **29°00′** | .4848 | .8746 | .5543 | **35°00′** | .5736 | .8192 | .7002 |
| 10 | .4874 | .8732 | .5581 | 10 | .5760 | .8175 | .7046 |
| 20 | .4899 | .8718 | .5619 | 20 | .5783 | .8158 | .7089 |
| 30 | .4924 | .8704 | .5658 | 30 | .5807 | .8141 | .7133 |
| 40 | .4950 | .8689 | .5696 | 40 | .5831 | .8124 | .7177 |
| 50 | .4975 | .8675 | .5735 | 50 | .5854 | .8107 | .7221 |

**Appendix** VALUES OF TRIGONOMETRIC RATIOS

| Degrees | sin | cos | tan | Degrees | sin | cos | tan |
|---|---|---|---|---|---|---|---|
| **36°00′** | .5878 | .8090 | .7265 | **42°00′** | .6691 | .7431 | .9004 |
| 10 | .5901 | .8073 | .7310 | 10 | .6713 | .7412 | .9057 |
| 20 | .5925 | .8056 | .7355 | 20 | .6734 | .7392 | .9110 |
| 30 | .5948 | .8039 | .7400 | 30 | .6756 | .7373 | .9163 |
| 40 | .5972 | .8021 | .7445 | 40 | .6777 | .7353 | .9217 |
| 50 | .5995 | .8004 | .7490 | 50 | .6799 | .7333 | .9271 |
| **37°00′** | .6018 | .7986 | .7536 | **43°00′** | .6820 | .7314 | .9325 |
| 10 | .6041 | .7969 | .7581 | 10 | .6841 | .7294 | .9380 |
| 20 | .6065 | .7951 | .7627 | 20 | .6862 | .7274 | .9435 |
| 30 | .6088 | .7934 | .7673 | 30 | .6884 | .7254 | .9490 |
| 40 | .6111 | .7916 | .7720 | 40 | .6905 | .7234 | .9545 |
| 50 | .6134 | .7898 | .7766 | 50 | .6926 | .7214 | .9601 |
| **38°00′** | .6157 | .7880 | .7813 | **44°00′** | .6947 | .7193 | .9657 |
| 10 | .6180 | .7862 | .7860 | 10 | .6967 | .7173 | .9713 |
| 20 | .6202 | .7844 | .7907 | 20 | .6988 | .7153 | .9770 |
| 30 | .6225 | .7826 | .7954 | 30 | .7009 | .7133 | .9827 |
| 40 | .6248 | .7808 | .8002 | 40 | .7030 | .7112 | .9884 |
| 50 | .6271 | .7790 | .8050 | 50 | .7050 | .7092 | .9942 |
| **39°00′** | .6293 | .7771 | .8098 | **45°00′** | .7071 | .7071 | 1.000 |
| 10 | .6316 | .7753 | .8146 | 10 | .7092 | .7050 | 1.006 |
| 20 | .6338 | .7735 | .8195 | 20 | .7112 | .7030 | 1.012 |
| 30 | .6361 | .7716 | .8243 | 30 | .7133 | .7009 | 1.018 |
| 40 | .6383 | .7698 | .8292 | 40 | .7153 | .6988 | 1.024 |
| 50 | .6406 | .7679 | .8342 | 50 | .7173 | .6967 | 1.030 |
| **40°00′** | .6428 | .7660 | .8391 | **46°00′** | .7193 | .6947 | 1.036 |
| 10 | .6450 | .7642 | .8441 | 10 | .7214 | .6926 | 1.042 |
| 20 | .6472 | .7623 | .8491 | 20 | .7234 | .6905 | 1.048 |
| 30 | .6494 | .7604 | .8541 | 30 | .7254 | .6884 | 1.054 |
| 40 | .6517 | .7585 | .8591 | 40 | .7274 | .6862 | 1.060 |
| 50 | .6539 | .7566 | .8642 | 50 | .7294 | .6841 | 1.066 |
| **41°00′** | .6561 | .7547 | .8693 | **47°00′** | .7314 | .6820 | 1.072 |
| 10 | .6583 | .7528 | .8744 | 10 | .7333 | .6799 | 1.079 |
| 20 | .6604 | .7509 | .8796 | 20 | .7353 | .6777 | 1.085 |
| 30 | .6626 | .7490 | .8847 | 30 | .7373 | .6756 | 1.091 |
| 40 | .6648 | .7470 | .8899 | 40 | .7392 | .6734 | 1.098 |
| 50 | .6670 | .7451 | .8952 | 50 | .7412 | .6713 | 1.104 |

## Appendix : VALUES OF TRIGONOMETRIC RATIOS

| Degrees | sin | cos | tan | Degrees | sin | cos | tan |
|---|---|---|---|---|---|---|---|
| **48°00′** | .7431 | .6691 | 1.111 | **54°00′** | .8090 | .5878 | 1.376 |
| 10 | .7451 | .6670 | 1.117 | 10 | .8107 | .5854 | 1.385 |
| 20 | .7470 | .6648 | 1.124 | 20 | .8124 | .5831 | 1.393 |
| 30 | .7490 | .6626 | 1.130 | 30 | .8141 | .5807 | 1.402 |
| 40 | .7509 | .6604 | 1.137 | 40 | .8158 | .5783 | 1.411 |
| 50 | .7528 | .6583 | 1.144 | 50 | .8175 | .5760 | 1.419 |
| **49°00′** | .7547 | .6561 | 1.150 | **55°00′** | .8192 | .5736 | 1.428 |
| 10 | .7566 | .6539 | 1.157 | 10 | .8208 | .5712 | 1.437 |
| 20 | .7585 | .6517 | 1.164 | 20 | .8225 | .5688 | 1.446 |
| 30 | .7604 | .6494 | 1.171 | 30 | .8241 | .5664 | 1.455 |
| 40 | .7623 | .6472 | 1.178 | 40 | .8258 | .5640 | 1.464 |
| 50 | .7642 | .6450 | 1.185 | 50 | .8274 | .5616 | 1.473 |
| **50°00′** | .7660 | .6428 | 1.192 | **56°00′** | .8290 | .5592 | 1.483 |
| 10 | .7679 | .6406 | 1.199 | 10 | .8307 | .5568 | 1.492 |
| 20 | .7698 | .6383 | 1.206 | 20 | .8323 | .5544 | 1.501 |
| 30 | .7716 | .6361 | 1.213 | 30 | .8339 | .5519 | 1.511 |
| 40 | .7735 | .6338 | 1.220 | 40 | .8355 | .5495 | 1.520 |
| 50 | .7753 | .6316 | 1.228 | 50 | .8371 | .5471 | 1.530 |
| **51°00** | .7771 | .6293 | 1.235 | **57°00′** | .8387 | .5446 | 1.540 |
| 10 | .7790 | .6271 | 1.242 | 10 | .8403 | .5422 | 1.550 |
| 20 | .7808 | .6248 | 1.250 | 20 | .8418 | .5398 | 1.560 |
| 30 | .7826 | .6225 | 1.257 | 30 | .8434 | .5373 | 1.570 |
| 40 | .7844 | .6202 | 1.265 | 40 | .8450 | .5348 | 1.580 |
| 50 | .7862 | .6180 | 1.272 | 50 | .8465 | .5324 | 1.590 |
| **52°00** | .7880 | .6157 | 1.280 | **58°00′** | .8480 | .5299 | 1.600 |
| 10 | .7898 | .6134 | 1.288 | 10 | .8496 | .5275 | 1.611 |
| 20 | .7916 | .6111 | 1.295 | 20 | .8511 | .5250 | 1.621 |
| 30 | .7934 | .6088 | 1.303 | 30 | .8526 | .5225 | 1.632 |
| 40 | .7951 | .6065 | 1.311 | 40 | .8542 | .5200 | 1.643 |
| 50 | .7969 | .6041 | 1.319 | 50 | .8557 | .5175 | 1.653 |
| **53°00′** | .7986 | .6018 | 1.327 | **59°00′** | .8572 | .5150 | 1.664 |
| 10 | .8004 | .5995 | 1.335 | 10 | .8587 | .5125 | 1.675 |
| 20 | .8021 | .5972 | 1.343 | 20 | .8601 | .5100 | 1.686 |
| 30 | .8039 | .5948 | 1.351 | 30 | .8616 | .5075 | 1.698 |
| 40 | .8056 | .5925 | 1.360 | 40 | .8631 | .5050 | 1.709 |
| 50 | .8073 | .5901 | 1.368 | 50 | .8646 | .5025 | 1.720 |

## Appendix : VALUES OF TRIGONOMETRIC RATIOS

| Degrees | sin | cos | tan | Degrees | sin | cos | tan |
|---|---|---|---|---|---|---|---|
| **60°00′** | .8660 | .5000 | 1.732 | **66°00′** | .9135 | .4067 | 2.246 |
| 10 | .8675 | .4975 | 1.744 | 10 | .9147 | .4041 | 2.264 |
| 20 | .8689 | .4950 | 1.756 | 20 | .9159 | .4014 | 2.282 |
| 30 | .8704 | .4924 | 1.767 | 30 | .9171 | .3987 | 2.300 |
| 40 | .8718 | .4899 | 1.780 | 40 | .9182 | .3961 | 2.318 |
| 50 | .8732 | .4874 | 1.792 | 50 | .9194 | .3934 | 2.337 |
| **61°00′** | .8746 | .4848 | 1.804 | **67°00′** | .9205 | .3907 | 2.356 |
| 10 | .8760 | .4823 | 1.816 | 10 | .9216 | .3881 | 2.375 |
| 20 | .8774 | .4797 | 1.829 | 20 | .9228 | .3854 | 2.394 |
| 30 | .8788 | .4772 | 1.842 | 30 | .9239 | .3827 | 2.414 |
| 40 | .8802 | .4746 | 1.855 | 40 | .9250 | .3800 | 2.434 |
| 50 | .8816 | .4720 | 1.868 | 50 | .9261 | .3773 | 2.455 |
| **62°00′** | .8829 | .4695 | 1.881 | **68°00′** | .9272 | .3746 | 2.475 |
| 10 | .8843 | .4669 | 1.894 | 10 | .9283 | .3719 | 2.496 |
| 20 | .8857 | .4643 | 1.907 | 20 | .9293 | .3692 | 2.517 |
| 30 | .8870 | .4617 | 1.921 | 30 | .9304 | .3665 | 2.539 |
| 40 | .8884 | .4592 | 1.935 | 40 | .9315 | .3638 | 2.560 |
| 50 | .8897 | .4566 | 1.949 | 50 | .9325 | .3611 | 2.583 |
| **63°00′** | .8910 | .4540 | 1.963 | **69°00** | .9336 | .3584 | 2.605 |
| 10 | .8923 | .4514 | 1.977 | 10 | .9346 | .3557 | 2.628 |
| 20 | .8936 | .4488 | 1.991 | 20 | .9356 | .3529 | 2.651 |
| 30 | .8949 | .4462 | 2.006 | 30 | .9367 | .3502 | 2.675 |
| 40 | .8962 | .4436 | 2.020 | 40 | .9377 | .3475 | 2.699 |
| 50 | .8975 | .4410 | 2.035 | 50 | .9387 | .3448 | 2.723 |
| **64°00′** | .8988 | .4384 | 2.050 | **70°00′** | .9397 | .3420 | 2.747 |
| 10 | .9001 | .4358 | 2.066 | 10 | .9407 | .3393 | 2.773 |
| 20 | .9013 | .4331 | 2.081 | 20 | .9417 | .3365 | 2.798 |
| 30 | .9026 | .4305 | 2.097 | 30 | .9426 | .3338 | 2.824 |
| 40 | .9038 | .4279 | 2.112 | 40 | .9436 | .3311 | 2.850 |
| 50 | .9051 | .4253 | 2.128 | 50 | .9446 | .3283 | 2.877 |
| **65°00′** | .9063 | .4226 | 2.145 | **71°00′** | .9455 | .3256 | 2.904 |
| 10 | .9075 | .4200 | 2.161 | 10 | .9465 | .3228 | 2.932 |
| 20 | .9088 | .4173 | 2.177 | 20 | .9474 | .3201 | 2.960 |
| 30 | .9100 | .4147 | 2.194 | 30 | .9483 | .3173 | 2.989 |
| 40 | .9112 | .4120 | 2.211 | 40 | .9492 | .3145 | 3.018 |
| 50 | .9124 | .4094 | 2.229 | 50 | .9502 | .3118 | 3.047 |

## Appendix : VALUES OF TRIGONOMETRIC RATIOS

| Degrees | sin | cos | tan | Degrees | sin | cos | tan |
|---|---|---|---|---|---|---|---|
| **72°00′** | .9511 | .3090 | 3.078 | **78°00′** | .9781 | .2079 | 4.705 |
| 10 | .9520 | .3062 | 3.108 | 10 | .9787 | .2051 | 4.773 |
| 20 | .9528 | .3035 | 3.140 | 20 | .9793 | .2022 | 4.843 |
| 30 | .9537 | .3007 | 3.172 | 30 | .9799 | .1994 | 4.915 |
| 40 | .9546 | .2979 | 3.204 | 40 | .9805 | .1965 | 4.989 |
| 50 | .9555 | .2952 | 3.237 | 50 | .9811 | .1937 | 5.066 |
| **73°00′** | ·9563 | .2924 | 3.271 | **79°00′** | .9816 | .1908 | 5.145 |
| 10 | .9572 | .2896 | 3.305 | 10 | .9822 | .1880 | 5.226 |
| 20 | .9580 | .2868 | 3.340 | 20 | .9827 | .1851 | 5.309 |
| 30 | .9588 | .2840 | 3.376 | 30 | .9833 | .1822 | 5.396 |
| 40 | .9596 | .2812 | 3.412 | 40 | .9838 | .1794 | 5.485 |
| 50 | .9605 | .2784 | 3.450 | 50 | .9843 | .1765 | 5.576 |
| **74°00′** | .9613 | .2756 | 3.487 | **80°00′** | .9848 | .1736 | 5.671 |
| 10 | .9621 | .2728 | 3.526 | 10 | .9853 | .1708 | 5.769 |
| 20 | .9628 | .2700 | 3.566 | 20 | .9858 | .1679 | 5.871 |
| 30 | .9636 | .2672 | 3.606 | 30 | .9863 | .1650 | 5.976 |
| 40 | .9644 | .2644 | 3.647 | 40 | .9868 | .1622 | 6.084 |
| 50 | .9652 | .2616 | 3.689 | 50 | .9872 | .1593 | 6.197 |
| **75°00′** | .9659 | .2588 | 3.732 | **81°00′** | .9877 | .1564 | 6.314 |
| 10 | .9667 | .2560 | 3.776 | 10 | .9881 | .1536 | 6.435 |
| 20 | .9674 | .2532 | 3.821 | 20 | .9886 | .1507 | 6.561 |
| 30 | .9681 | .2504 | 3.867 | 30 | .9890 | .1478 | 6.691 |
| 40 | .9689 | .2476 | 3.914 | 40 | .9894 | .1449 | 6.827 |
| 50 | .9696 | .2447 | 3.962 | 50 | .9899 | .1421 | 6.968 |
| **76°00′** | .9703 | .2419 | 4.011 | **82°00′** | .9903 | .1392 | 7.115 |
| 10 | .9710 | .2391 | 4.061 | 10 | .9907 | .1363 | 7.269 |
| 20 | .9717 | .2363 | 4.113 | 20 | .9911 | .1334 | 7.429 |
| 30 | .9724 | .2334 | 4.165 | 30 | .9914 | .1305 | 7.596 |
| 40 | .9730 | .2306 | 4.219 | 40 | .9918 | .1276 | 7.770 |
| 50 | .9737 | .2278 | 4.275 | 50 | .9922 | .1248 | 7.953 |
| **77°00′** | .9744 | .2250 | 4.331 | **83°00′** | .9925 | .1219 | 8.144 |
| 10 | .9750 | .2221 | 4.390 | 10 | .9929 | ·1190 | 8.345 |
| 20 | .9757 | .2193 | 4.449 | 20 | ·9932 | .1161 | 8.556 |
| 30 | .9763 | .2164 | 4.511 | 30 | .9936 | .1132 | 8.777 |
| 40 | .9769 | .2136 | 4.574 | 40 | .9939 | .1103 | 9.010 |
| 50 | .9775 | 2108 | 4·638 | 50 | .9942 | .1074 | [illegible] |

## Appendix : VALUES OF TRIGONOMETRIC RATIOS

| Degrees | sin | cos | tan. | Degrees | sin | cos | tan |
|---|---|---|---|---|---|---|---|
| **84°00′** | .9945 | .1045 | 9.514 | **87°00′** | .9986 | .0523 | 19.08 |
| 10 | .9948 | .1016 | 9.788 | 10 | .9988 | .0494 | 20.21 |
| 20 | .9951 | .0987 | 10.08 | 20 | .9989 | .0465 | 21.47 |
| 30 | .9954 | .0958 | 10.39 | 30 | .9990 | .0436 | 22.90 |
| 40 | .9957 | .0929 | 10.71 | 40 | .9992 | .0407 | 24.54 |
| 50 | .9959 | .0901 | 11.06 | 50 | .9993 | .0378 | 26.43 |
| **85°00** | 9962 | .0872 | 11.43 | **88°00′** | .9994 | .0349 | 28.64 |
| 10 | .9964 | .0843 | 11.83 | 10 | .9995 | .0320 | 31.24 |
| 20 | .9967 | .0814 | 12.25 | 20 | .9996 | .0291 | 34.37 |
| 30 | .9969 | .0785 | 12.71 | 30 | .9997 | .0262 | 38.19 |
| 40 | .9971 | .0756 | 13.20 | 40 | .9997 | .0233 | 42.96 |
| 50 | .9974 | .0727 | 13.73 | 50 | .9998 | .0204 | 49.10 |
| **86°00′** | .9976 | .0698 | 14.30 | **89°00′** | .9998 | .0175 | 57.29 |
| 10 | .9978 | .0669 | 14.92 | 10 | .9999 | .0145 | 68.75 |
| 20 | .9980 | .0640 | 15.60 | 20 | .9999 | .0116 | 85.94 |
| 30 | .9981 | .0610 | 16.35 | 30 | 1.0000 | .0087 | 114.6 |
| 40 | .9983 | .0581 | 17.17 | 40 | 1.0000 | .0058 | 171.9 |
| 50 | .9985 | .0552 | 18.07 | 50 | 1.0000 | .0029 | 343.8 |
| | | | | **90°00′** | 1.0000 | 0.0000 | |

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